

## Ropelength criticality

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Revised: March 23, 2011

**Abstract** The *ropelength problem* asks for the minimum-length configuration of a knotted tube embedded with fixed diameter. The core curve of such a tube is called a tight knot, and its length is a knot invariant measuring complexity. In terms of the core curve, the thickness constraint has two parts: an upper bound on curvature and a self-contact condition.

We give a set of necessary and sufficient conditions for criticality with respect to this constraint, based on a version of the Kuhn–Tucker theorem that we established in previous work. The key technical difficulty is to compute the derivative of thickness under a smooth perturbation. This is accomplished by writing thickness as the minimum of a  $C^1$ -compact family of smooth functions in order to apply a theorem of Clarke. We give a number of applications, including a classification of critical curves with no self-contacts (constrained by curvature alone), a characterization of helical segments in tight links, and an explicit but surprisingly complicated description of tight clasps.

**Keywords** ropelength, ideal knot, tight knot, constrained minimization, Kuhn–Tucker theorem, simple clasp, Clarke gradient

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*Unlike the classical machine that is composed of well-defined parts  
that interact according to well-understood rules (gears and cogs),  
the sliding interaction of two ropes under tension is extraordinary and interactive,  
with tension, topology, and the system providing the form which finally results.*

—Louis H. Kauffman, *Knots and Physics*, 1992

## 1 Introduction

Our goal in this paper is to investigate what shape a knot or link attains when it is tied in rope of a given thickness and then pulled tight. Ignoring elastic deformations within the rope, we formulate this as the *ropelength problem*: to minimize the length of a knot or link  $L$  in euclidean space subject to the condition that it remain one unit thick. Although there are many equivalent formulations [CKS02, GM99a] of this thickness constraint, perhaps the most elegant simply requires that the *reach* of  $L$  be at least  $1/2$ . Here, following Federer, the reach of  $L$  is the supremal  $r \geq 0$  such that every point in space within distance  $r$  of  $L$  has a unique nearest point on  $L$ . Any curve of positive reach is  $C^{1,1}$ , that is, its unit tangent vector is a Lipschitz function of arclength.

In an earlier paper [CFK<sup>+</sup>06], we studied a simplified version, the Gehring link problem. Here the thickness constraint is replaced by the weaker requirement that the *Gehring thickness* – the minimal distance between different components of the link – is at least 1. Thinking of the components again as strands of rope of diameter 1, this means that different strands cannot overlap, but each strand can pass through itself. Our balance criterion [CFK<sup>+</sup>06] for the Gehring problem made precise the intuition that, in a critical configuration for a link  $L$ , the tension forces seeking to minimize length must be balanced by contact forces. More precisely, we defined a *strut* to be a pair of points on different components at distance exactly 1. The balance criterion says that  $L$  is critical if and only if there is a nonnegative measure on the set of struts, thought of as a system of compression forces, which balances the curvature vector field of  $L$ .

The proof was based on two main technical tools. First, we used Clarke’s theorem on the derivatives of “min-functions” [Cla75] to compute the directional derivative of the Gehring thickness with respect to a smooth deformation of  $L$ . This is possible because the Gehring thickness may be expressed as the minimum of the compact family of smooth functions given by the distances between all pairs of points lying on different components of  $L$ . Second, we proved a new version of the Kuhn–Tucker theorem on extrema of functionals subject to convex constraints, similar in spirit to a version by Luenberger [Lue69], but giving necessary and sufficient conditions for a strong form of criticality.

In the present paper we adopt the same general approach to develop a criticality theory for the (technically much more difficult) ropelength problem. Again the main point is to express the thickness as the minimum of a collection of smooth functions, here the union of two disjoint compact subfamilies. The first subfamily is indexed by *all* pairs of points of the link  $L$ , and essentially measures the distance, but is modified so as to ignore pairs lying close to one another along a single component. This yields a  $C^1$ -compact family of functions indexed by  $L \times L$ . The second subfamily controls

the curvature of  $L$ , but its construction is complicated by the fact that  $L$  need not be  $C^2$ . Nevertheless, since any thick curve is  $C^{1,1}$ , by Rademacher's theorem  $L$  admits an osculating circle almost everywhere. Our second subfamily is indexed by the closure  $\overline{\text{Osc}L}$  of the set of these osculating circles in the space of all pointed circles in  $\mathbb{R}^3$ ; the functions simply measure the radius of each circle.

Proceeding in this way, we formulate and prove our first main result – the General Balance Criterion of Theorem 4.15 – which gives a necessary and sufficient condition for a link to be critical for length under the thickness constraint. As in the Gehring case, the condition requires the existence of a certain measure balancing the curvature of  $L$ , this time the sum of the strut measure and a *kink measure* on the space  $\overline{\text{Osc}L}$  of circles. In particular, in the case when there are no kinks, we recover the criticality criterion of Schuricht and von der Mosel [SvdM04], who discussed tight knots where the curvature constraint is nowhere active.

Our analysis also applies to the case where, in addition to the thickness constraint, the radius of curvature of the curve is constrained to be at least  $\sigma$ , a parameter giving the *stiffness* of the link. (Here we take  $\sigma \geq 1/2$ , with  $\sigma = 1/2$  corresponding to the ordinary ropelength problem.)

The General Balance Criterion can be applied directly to curves without kinks; for example we classify curves with struts in one-to-one contact as double helices. The kink measure, on the other hand, is a bit arcane and can be difficult to work with: in general,  $L$  is no smoother than  $C^{1,1}$ , so the space  $\overline{\text{Osc}L}$  may be an unruly subspace of the normal bundle over  $L$ . For a  $C^2$  link, of course, the kink measure reduces to a measure along  $L$ , but unfortunately, the only known example of a tight link which is  $C^2$  is the round circle, the ropelength-minimizing unknot. On the other hand, all known explicit examples of tight links [CKS02, CFK<sup>+</sup>06] are piecewise  $C^2$ .

With a view towards the fact that other tight links (say, the tight trefoil knot) may not even be piecewise  $C^2$ , in Section 5, we impose the even milder smoothness assumption of *regulated kinks*. We conjecture that all critical links have regulated kinks, but an answer to this question seems far beyond our current understanding. For links with regulated kinks, we derive successively nicer forms of our Balance Criterion, concluding with Theorem 5.10, our second main result. It says the kink measure can be described by a scalar *kink tension function* – or equivalently, by a *virtual tangent* vector – along the curve. As an example, we use this theorem to classify all strut-free arcs in critical curves.

At the end of the paper, we apply our Balance Criterion to describe the ropelength-critical symmetric clasps. A curious feature of these clasps – whose analysis is based on the discussion in [CFK<sup>+</sup>06, Sect. 9] – is the presence of a gap between the tips of the two components. In other words, there is a small cavity between two tight ropes of circular cross-section linked in this way.

## 2 Curves and their smoothness classes

We consider generalized links, which may include arc components with constrained boundary points; our links are always  $C^1$  but not necessarily  $C^2$ .

A  $C^1$  **curve**  $L$  will mean a compact 1-dimensional  $C^1$ -submanifold with boundary embedded in  $\mathbb{R}^3$ . (For us, manifold will always mean manifold with boundary.) The curve  $L$  is thus a finite union of components, each a circle or an arc (compact interval). Our results are independent of orientation, but for convenience (especially in talking about derivatives) we fix an orientation on each component. The Euclidean metric on  $\mathbb{R}^3$  pulls back to give a Riemannian metric on  $L$ ; in particular at each point  $x \in L$  we have (up to orientation) a unit tangent vector  $T(x)$ . The orientation induces a sign  $\pm 1$  on each endpoint  $p \in \partial L$ ; we write  $\pm T$  for the outward tangent vector.

Each arc or circle component of length  $\ell$  is of course isometric to  $[0, \ell]$  or  $\mathbb{R}/\ell\mathbb{Z}$ , respectively. Writing  $M$  for the disjoint union of these intervals or circles, the isometry  $M \rightarrow L \subset \mathbb{R}^3$  is simply the arclength parametrization of  $L$ ; we implicitly identify  $M$  with  $L$  and thus identify the arclength parametrization with the inclusion map  $\gamma: L \hookrightarrow \mathbb{R}^3$ . While  $L$  inherits only a  $C^1$  structure directly as a submanifold of  $\mathbb{R}^3$ , it does have a natural  $C^\infty$  structure: the standard  $C^\infty$  structure on  $\mathbb{R}$  induces one on  $M$  and thus on  $L$  via the isometry.

All standard smoothness classes of functions on  $L$  are obtained via this identification. In particular, given a (vector-valued) function  $f$  on  $L$ , at any  $x \in L$  we write

$$\frac{\partial f(x)}{\partial x} := f'(x) := D_x f(T(x))$$

for the arclength derivative of  $f$  (if this exists). Thus a  $C^1$  function  $f$  has a continuous derivative  $f'$ ; for example  $\gamma'(x) = T(x)$ .

It is a standard and straightforward fact that no regular (meaning immersive) parametrization of a Lipschitz curve is smoother than its arclength parametrization. Thus when we talk about the smoothness of  $L$  we mean the smoothness of the inclusion  $\gamma$ . For any  $C^1$  curve  $L$ , we let  $E = E_L \subset L$  denote the set of points at which  $L$  (meaning  $\gamma$ ) is twice differentiable. (At an endpoint  $x \in \partial L$  we of course require only a one-sided second derivative.) No regular parametrization has a second derivative at any point of  $L \setminus E$ . For  $x \in E$ , we write  $\kappa(x) := T'(x) = \gamma''(x)$  for the curvature vector. Recall that by Rademacher's Theorem  $E$  has full measure if  $L$  is  $C^{1,1}$  (meaning  $T$  is Lipschitz).

**Lemma 2.1** *Suppose  $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a  $C^2$  diffeomorphism and  $L \subset \mathbb{R}^3$  is a  $C^1$  curve. Then its image  $\phi L$  is a  $C^1$  curve with  $E_{\phi L} = \phi E_L$ .*

*Proof* If  $\gamma$  is the arclength parametrization of  $L$ , then  $\phi \circ \gamma$  is a regular parametrization of  $\phi L$ . Since its second derivative exists at all points of  $\phi E_L$  we have  $\phi E_L \subset E_{\phi L}$ . The reverse inclusion follows by considering  $L$  as the image of  $\phi L$  under  $\phi^{-1}$ .  $\square$

Suppose we have a  $C^2$ -smooth vector-valued function  $f: \mathbb{R}^3 \rightarrow V$  on space. Its restriction  $f|_L = f \circ \gamma$  to  $L$  is  $C^1$  with  $f'(x) = D_x f(T(x))$  by the chain rule. For  $x \in E$ , the second arclength derivative along  $L$  also exists:

$$f''(x) = D_x^2 f(T(x), T(x)) + D_x f(\kappa(x)).$$

We say a sequence  $L_1, L_2, \dots$  of  $C^1$  curves **converges in the  $C^1$  topology** to a  $C^1$  curve  $L$  if there are  $C^1$  immersions  $\gamma_i: L \rightarrow \mathbb{R}^3$  with images  $\gamma_i(L) = L_i$  such that

the maps  $\gamma_i$  converge in  $C^1$  to the inclusion map  $\gamma$ . Of course each  $\gamma_i$  has a unique reparametrization  $\gamma_i \circ \phi_i$  with locally constant speed (that is, constant speed on each component). Since these also converge to  $\gamma$ , we usually assume each  $\gamma_i$  has locally constant speed.

## 2.1 Regulated functions and functions of bounded variation

Two classes which will be of importance later are regulated functions and functions of bounded variation. While these are often discussed for functions on an interval in  $\mathbb{R}$ , it is equivalent to define them on Riemannian 1-manifolds; in our context we speak of submanifolds  $M$  of a  $C^1$  curve  $L$ . (Any 1-manifold is a countable union of components, each a circle or an open, half-open or compact interval.) Note that a submanifold  $M \subset L$  with empty boundary is exactly an open subset  $U \subset L \setminus \partial L$ .

Let  $M \subset L$  be a submanifold of a  $C^1$  curve. A **regulated function** on  $M$  is a function  $f: D \rightarrow \mathbb{R}^n$  defined on a dense subset  $D \subset M$  whose one-sided limits exist at every  $x \in M$ . An interior point  $x \in M \setminus \partial M$  is called a **jump point** of  $f$  if  $f(x-) \neq f(x+)$ . For  $\varepsilon > 0$  we let  $J_\varepsilon$  denote the set on which the jump is large:

$$J_\varepsilon(f) := \{x \in M \setminus \partial M : |f(x-) - f(x+)| \geq \varepsilon\}.$$

If  $M$  is compact then  $J_\varepsilon$  is finite; for any  $M$  it follows that  $J_\varepsilon$  is countable and closed in  $M$  (though not necessarily in  $L$ ). The union  $J = J(f) := \bigcup J_\varepsilon(f) \subset M$  is the countable set of all jump points (which may of course be dense). Let  $\bar{f}: M \rightarrow \mathbb{R}^n$  denote any function such that  $\bar{f}(x) \in \{f(x-), f(x+)\}$  for each  $x$ . (Note that  $\bar{f} = f$  at all but countably many points of  $D$  – a statement which is vacuous if  $D$  is countable.) Then  $\bar{f}$  is continuous on  $M \setminus J$  but has a jump discontinuity at each  $x \in J$ . The following lemma is then immediate:

**Lemma 2.2** *Let  $f$  be a regulated function on  $M$ . Consider the smoothings  $f_\varepsilon := \bar{f} * \phi_\varepsilon$  obtained by convolution with a sequence of mollifiers. (Here  $f_\varepsilon$  is defined away from an  $\varepsilon$ -neighborhood of  $\partial M$ .) For any  $x \in M \setminus (\partial M \cup J)$ , the continuity of  $\bar{f}$  at  $x$  implies that  $f_\varepsilon(x) \rightarrow \bar{f}(x)$ . In particular we have this pointwise convergence at all but countably many points of  $M$ .  $\square$*

The following lemma gives another sense in which regulated functions are close to being continuous.

**Lemma 2.3** *Let  $f$  be a real-valued regulated function on  $M$ . For any  $b > a \in \mathbb{R}$  there is an open subset  $U \subset M$  such that*

$$\{x : \bar{f}(x) \geq b\} \subset U \subset \{x : \bar{f}(x) > a\}.$$

*Proof* We may replace  $M$  by the open subset  $M \setminus J_{b-a}$  to assume that all jumps of  $\bar{f}$  are smaller than  $b - a$ . Thus any  $x$  with  $\bar{f}(x) \geq b$  has a neighborhood  $U_x$  on which  $f > a$ . We simply let  $U$  be the union of all such neighborhoods  $U_x$ .  $\square$

We will say that an absolutely continuous function  $g: M \rightarrow \mathbb{R}^n$  has **regulated derivative** if its arclength derivative  $g'$  (which is defined almost everywhere) is regulated. Note that in this case the mean value theorem implies that  $g'(x\pm)$  are the one-sided derivatives of  $g$ , so  $g'(x)$  exists if and only if  $g'(x+) = g'(x-)$ .

**Lemma 2.4** *Let  $f: (a, b) \rightarrow (c, d)$  be a  $C^{1,1}$  diffeomorphism with  $1/2 \leq f' \leq 1$ . Its inverse  $g$  is also  $C^{1,1}$  with  $1 \leq g' \leq 2$ . Furthermore  $f$  has regulated second derivative if and only if  $g$  does.*

*Proof* The chain rule gives  $g'(f(x)) = 1/f'(x)$ ; therefore if  $f'$  is  $L$ -Lipschitz then  $g'$  is  $8L$ -Lipschitz. The second derivative  $g''$  exists almost everywhere and from the formula  $g''(f(x)) = -f''(x)/f'(x)^3$  we see that it has a one-sided limit at  $f(x)$  if and only if  $f''$  has a one-sided limit at  $x$ .  $\square$

Now consider the space  $BV(M, \mathbb{R}^n)$  of functions of bounded (essential) variation, again on a submanifold  $M \subset L$  of a  $C^1$  curve. For  $k \geq 1$  we write  $W^{k,BV}(M, \mathbb{R}^n)$  for the Sobolev space of functions whose  $k^{\text{th}}$  (distributional) derivatives (with respect to arclength) lie in  $BV(M, \mathbb{R}^n)$ . We write  $BV_{\text{loc}}(M, \mathbb{R}^n)$  for the space of functions with locally bounded variation in  $M$ , and similarly for  $W_{\text{loc}}^{k,BV}(M, \mathbb{R}^n)$ . We recall a few facts about BV functions.

- Any  $f \in BV_{\text{loc}}(M, \mathbb{R}^n)$  (after modification on a set of measure zero) is regulated, i.e., has only jump discontinuities.
- We have  $f \in BV_{\text{loc}}(M, \mathbb{R}^n)$  if and only if its distributional derivative is a vector-valued Radon measure (with atoms at the jumps of  $f$ ).
- Any function  $g \in W_{\text{loc}}^{1,BV}(M, \mathbb{R}^n)$  is locally Lipschitz continuous.

### 3 Thickness, reach and curvature

Let  $L$  be a  $C^1$  curve in  $\mathbb{R}^3$ . At any interior point  $x \in L$ , the **tangent cone**  $T_x L$  is the line through  $x$  tangent to  $L$ . At an endpoint  $x \in \partial L$  of an arc component,  $T_x L$  is the (inward) tangent ray. The **normal cone**  $N_x L$  is

$$N_x L := \{p \in \mathbb{R}^3 : \langle p - x, q - x \rangle \leq 0 \text{ for all } q \in T_x L\}.$$

At an interior point this is the normal plane, while at an endpoint  $x \in \partial L$  it is a closed halfspace. (These cones are the translates by the base point  $x$  of the corresponding cones given by [Fed59] for general closed subsets of  $\mathbb{R}^n$ .)

If  $p \notin N_x L$ , then there are points near  $x$  in  $L$  which are closer to  $p$ . Thus if  $(x, y)$  is a local minimum for  $|x - y|$  on  $L \times L$  (away from the diagonal), then  $(x, y)$  is a critical pair in the following sense:

**Definition 3.1** A pair of distinct points  $x, y \in L$  is a **critical pair** if  $x \in N_y L$  and  $y \in N_x L$ . We denote the set of all critical pairs by  $\text{Crit}(L)$ .

Federer's definition [Fed59] of reach can be phrased as follows:

**Definition 3.2** Given a link (or indeed any closed set)  $L \subset \mathbb{R}^3$ , its **medial axis** is the set of points  $p \in \mathbb{R}^3$  for which the nearest point  $x \in L$  is not unique. The **reach** of  $L$ ,  $\text{reach}(L)$ , is the distance from  $L$  to its medial axis.

Of course, a closed subset  $L \subset \mathbb{R}^3$  has infinite reach if and only if it is convex. For curves, this means  $\text{reach}(L) = \infty$  if and only if  $L$  is a connected straight arc. We will often exclude this trivial case, for instance when discussing derivatives of reach.

The following alternate characterization of reach is an immediate corollary of [Fed59, Theorem 4.8].

**Lemma 3.3** *If  $L$  is a  $C^1$  curve in  $\mathbb{R}^3$  then the reach of  $L$  equals the infimal  $r > 0$  such that there exist  $x \neq y \in L$  and  $p \in N_x L$  with  $|p - x| = r = |p - y|$ .*  $\square$

For distinct points  $x, y \in L$ , let  $C(x, y)$  denote the circle through  $y$  tangent to  $L$  at  $x$ . By plane geometry, its radius is

$$\frac{|x - y|}{2 \cos \psi(x, y)} =: r(x, y),$$

where  $\psi(x, y) \in [0, \pi/2]$  denotes the angle between the normal plane to  $L$  at  $x$  and the segment  $xy$ . (Our notation here suppresses the dependence on  $L$ , in particular on  $T_x L$ .)

To properly handle endpoints of generalized links, we also need variants of these functions. So consider now circles in the plane of  $T_x L$  and  $y$ , passing through  $x$  and  $y$ . Let  $C^*(x, y)$  denote the smallest such circle whose center lies in  $N_x L$ . Then  $C^*(x, y) = C(x, y)$  except when  $x \in \partial L$  and  $y \in N_x L$ , in which case  $C^*(x, y)$  is a circle with diameter  $xy$ . The radius of  $C^*(x, y)$  is

$$\frac{|x - y|}{2 \cos \psi^*(x, y)} =: r^*(x, y) \leq r(x, y),$$

where  $\psi^*(x, y) \in [0, \pi/2]$  denotes the angle at  $x$  between  $N_x L$  and the segment  $xy$ . Thus  $\psi^* = 0$  for  $y \in N_x L$  and  $\psi^* = \pi/2$  for  $y \in T_x L$ . Furthermore  $\psi^*(x, y) = \psi(x, y)$  if  $x$  is an interior point.

Lemma 3.3 can now be rephrased as follows:

**Corollary 3.4** *If  $L$  is a  $C^1$  curve in  $\mathbb{R}^3$  then*

$$\text{reach}(L) = \inf_{x \neq y \in L} r^*(x, y) = \min \left( \inf_{x \neq y \in L} r(x, y), \inf_{\substack{x \neq y \in L \\ x \in \partial L}} r^*(x, y) \right).$$

*Proof* Any point  $p \in N_x L$  as in Lemma 3.3 is the center of a circle through  $x$  and  $y$ ; hence  $|p - x| \geq r^*(x, y)$ . Conversely, the center of any  $C^*(x, y)$  is such a point  $p$ . This gives the first equality. The second follows from the fact that  $r^*(x, y) \leq r(x, y)$  with equality unless  $x \in \partial L$ .  $\square$

(For closed curves, this was also the first statement in [CKS02, Lemma 1]. The proof of the later parts of that lemma should have been more careful about the treatment of points where  $L$  is not twice differentiable.)

For any  $C^1$  link  $L$ , the angles  $\psi$  and  $\psi^*$  extend continuously to the diagonal, since  $\lim_{y \rightarrow x} \psi(x, y) = \pi/2 = \lim_{y \rightarrow x} \psi^*(x, y)$ . But without additional smoothness of  $L$ , the

functions  $r$  and  $r^*$  do not extend. For smooth curves, of course, it is a standard fact that as  $y \rightarrow x$ , the circles tangent at  $x$  through  $y$  approach the osculating circle at  $x$ . For completeness, we verify that the existence of a second derivative at  $x$  is sufficient for this:

**Lemma 3.5** *Suppose  $L$  is a  $C^1$  curve with curvature vector  $\kappa$  at a point  $x \in E_L$ . Then*

$$\lim_{y \rightarrow x} r(x, y) = \lim_{y \rightarrow x} r^*(x, y) = 1/|\kappa|.$$

*Proof* First note that for  $y$  sufficiently near  $x$ , we have  $y \notin N_x L$  so  $\psi^*(x, y) = \psi(x, y)$  and thus  $r^*(x, y) = r(x, y)$ . Assume  $x = 0 \in \mathbb{R}^3$  and let  $\gamma$  be an arclength parametrization around  $x$  so

$$\gamma(0) = 0, \quad \gamma'(0) = T = T(x), \quad \gamma''(0) = \kappa.$$

Taylor's theorem implies that

$$\gamma(s) = sT + \frac{s^2}{2}\kappa + o(s^2).$$

For  $y = \gamma(s)$ , we can compute  $\psi$  from the equation  $|T \times y| = |y| \cos \psi(x, y)$ . We get

$$r(x, y) = \frac{|\gamma(s)|^2}{2|T \times \gamma(s)|} = \frac{s^2 + o(s^3)}{|\kappa|s^2 + o(s^2)} = 1/|\kappa| + o(1).$$

□

**Lemma 3.6** *Suppose a  $C^1$  curve  $L$  is twice differentiable at  $x \in E_L$ , and suppose  $y \in L \setminus N_x L$ . Fix the orientation at  $x$  such that  $\langle T(x), y - x \rangle > 0$ . If  $r(x, y) < \infty$ , then the partial derivative  $\partial/\partial x r(x, y)$  exists, with*

$$\frac{\partial r}{\partial x} \leq (r(x, y)|\kappa(x)| - 1) \tan \psi(x, y).$$

*Proof* From plane geometry, the rotation speed of the vector  $x - y$  is

$$\left| \frac{\partial}{\partial x} \left( \frac{x - y}{|x - y|} \right) \right| = \frac{1}{2r(x, y)}.$$

The normal plane  $N_x L$  of course turns at rate  $|\kappa(x)|$ . Comparing these rates gives

$$-\frac{1}{2r(x, y)} - |\kappa(x)| \leq \frac{\partial \psi(x, y)}{\partial x} \leq -\frac{1}{2r(x, y)} + |\kappa(x)|.$$

On the other hand differentiating the definition of  $r$  gives

$$\frac{\partial r(x, y)}{\partial x} = -\frac{1}{2} \tan \psi + r \tan \psi \frac{\partial \psi}{\partial x}.$$

The desired inequality follows at once. □



### 3.1 Penalized distance

In order to apply Clarke's theorem to compute the derivative of  $\text{reach}(L)$  under a smooth deformation of  $L$ , we must express the reach of  $L$  as the minimum of a  $C^1$ -compact family of functions. For a closed  $C^2$  curve  $L$ , we could simply extend  $r$  continuously to the diagonal  $x = y$  by the last lemma, and get a compact family parametrized by  $L \times L$ . The three-point curvature of [GM99a] gives another approach that could also be used for  $C^2$  curves. Unfortunately, the examples of [CKS02] show that even ropelength minimizers may fail to be  $C^2$ .

On the other hand by [CKS02, Lemma 4], the reach condition implies that  $L$  is  $C^{1,1}$ , hence twice differentiable almost everywhere by Rademacher's theorem; this turns out to be enough to make Clarke's theorem work using a more technical approach, as follows. First, if the infimal  $r$  is achieved, then it is achieved for a critical pair  $(x, y)$ , where  $r = |x - y|/2$ . To avoid the problem that the infimum might also be achieved at noncritical pairs, we define a penalized distance function that achieves its minimum only on critical pairs. Second, if the infimal  $r$  is not achieved, then it is approached in the limit as  $y \rightarrow x$ . Intuitively, this should happen at a point of maximum curvature, but in fact  $L$  might not even be twice differentiable at the limit point. To handle this limiting behavior near the diagonal, we will look at the set of osculating circles (at points where  $L$  is twice differentiable) and compactify it within the space of all pointed circles in space.

**Definition 3.7** Given a link  $L$ , the **penalized distance** between two distinct points  $x, y \in L$  is

$$\text{pd}(x, y) := |x - y| \sec^2 \psi(x, y) = 2r(x, y) \sec \psi(x, y).$$

For  $y = x$ , we set  $\text{pd}(x, x) = \infty$ . When we want to emphasize the dependence on  $L$ , we will write  $\text{pd}^L(x, y)$ . Similarly the **penalized endpoint distance** is

$$\text{pd}^*(x, y) := |x - y| \sec^2 \psi^*(x, y) = 2r^*(x, y) \sec \psi^*(x, y) \leq \text{pd}(x, y).$$

For  $y = x$ , we set  $\text{pd}^*(x, x) = \infty$ . Of course  $\text{pd}^*(x, y) = \text{pd}(x, y)$  except when  $x \in \partial L$ .

**Lemma 3.8** *Given a link  $L$  of positive reach, the penalized distance is a continuous function from  $L \times L$  to  $(0, \infty]$ . Similarly, the penalized endpoint distance is continuous on  $\partial L \times L$ .*

*Proof* First, we note that the angle  $\psi(x, y)$  (extended to be  $\pi/2$  on the diagonal  $x = y$ ) is continuous. The formula for  $\text{pd}(x, y)$  shows it shares this continuity away from the diagonal  $x = y$ . But we also have continuity on the diagonal, since  $r \geq \text{reach}(L) > 0$ , while  $\psi$  approaches  $\pi/2$  as  $(x, y) \rightarrow (z, z)$ .

On the other hand the penalized endpoint distance  $\text{pd}^*(x, y)$  is merely lower semi-continuous, since it equals  $\text{pd}(x, y)$  away from endpoints  $x \in \partial L$  but can jump down there. But the continuity claimed here is easy: for fixed  $x \in \partial L$ , the angle  $\psi^*(x, y)$  is continuous in  $y$ , and the rest follows as above.  $\square$

**Lemma 3.9** *Suppose  $0 < \text{reach}(L) < \infty$ . We have  $\text{pd}^*(x, y) \geq 2\text{reach}(L)$  for all  $x, y \in L$ ; equality can hold only if  $x, y$  is a critical pair.*

*Proof* Clearly  $\text{pd}^*(x, y) \geq 2r^*(x, y)$ , with equality only when  $\psi^*(x, y) = 0$ , that is, when  $y \in N_x L$ . Since  $r^*(x, y) \geq \text{reach}(L)$  by Corollary 3.4, it only remains to show that  $x \in N_y L$  in the case  $\text{pd}^*(x, y) = 2\text{reach}(L)$ . If not, there is a tangent vector  $T$  to  $L$  at  $y$  such that  $\langle x - y, T \rangle > 0$ . The directional derivative of  $|x - y|$  in the direction  $T$  is negative; since  $\psi^*(x, y) = 0$ , the directional derivative of  $\text{pd}^*(x, y)$  is the same negative value, contradicting the fact that  $\text{pd}^*(x, y) = \text{reach}(L)$  is a minimum.  $\square$

### 3.2 Osculating circles

Now we consider the space  $C_3$  of all oriented pointed circles in  $\mathbb{R}^3$ , which we identify with  $\mathbb{R}^3 \times TS^2$  by taking  $(p, C)$  to correspond to  $(p, T, \kappa) \in \mathbb{R}^3 \times TS^2$ , where  $T$  is the oriented unit tangent to  $C$  at  $p$  and  $\kappa$  is its curvature vector there. Let  $R(p, T, \kappa) := 1/|\kappa| \in (0, \infty]$  be the radius function. In this formulation the circles  $C$  may degenerate to lines, with  $\kappa = 0$  and  $R = \infty$ . Let  $\Pi$  be the projection  $\Pi: (p, C) \mapsto p$ .

Given a  $C^{1,1}$  link  $L$ , the set  $E$  on which the second derivative exists has full measure. Note that the minimal Lipschitz constant  $\text{Lip}(T)$  for the tangent vector as a function of arclength is exactly  $\sup_E |\kappa|$ . We let  $\text{Osc} L \subset C_3$  be the set of all osculating circles:

$$\text{Osc} L := \{(x, T(x), \kappa(x)) : x \in E\} \subset C_3.$$

Its closure  $\overline{\text{Osc} L}$  is a compact subset of  $C_3$  since  $|\kappa|$  is bounded on  $E$ . Note that  $T = T(x)$  for any  $(x, T, \kappa) \in \overline{\text{Osc} L}$ , while of course  $\kappa \perp T$  is some normal vector; thus we can view  $\overline{\text{Osc} L}$  as a subset of the normal bundle to  $L$ .

For  $x \in L$ , we set  $\overline{\text{Osc} L}_x := \overline{\text{Osc} L} \cap \Pi^{-1}\{x\}$ . Since  $E \subset L$  is dense, it follows that  $\overline{\text{Osc} L}_x$  is nonempty for every point  $x \in L$ . Thus for  $x \in L$  we may define

$$\rho(x) := \min_{\overline{\text{Osc} L}_x} R = \left( \lim_{E \ni y \rightarrow x} |\kappa(y)| \right)^{-1}.$$

Note that  $\rho$  is essentially a Clarke upper derivative of the tangent vector  $T$ . Clearly  $\rho$  is lower semicontinuous. For  $x \in E$  we have  $\rho(x) \leq 1/|\kappa(x)|$ , but equality might not hold.

**Lemma 3.10** *If  $L$  is a  $C^{1,1}$  curve and  $c \in \overline{\text{Osc} L}$  then  $R(c) \geq \text{reach}(L)$ .*

*Proof* By continuity of  $R$ , it is enough to prove this for osculating circles  $c \in \text{Osc} L$ . There it follows immediately from Corollary 3.4 and Lemma 3.5.  $\square$

**Lemma 3.11** *If  $r(x, y) = \text{reach}(L)$  with  $y \notin N_x L$ , then  $\rho(x) = \text{reach}(L)$ .*

*Proof* If not, we have  $r(x, y) < \rho(x)$ , in which case by lower semicontinuity of  $\rho$  there is a neighborhood  $U$  of  $x$  in  $L$  such that  $r(x', y) < \rho(x')$  for  $x' \in U$ . At any  $x' \in E \cap U$  we have  $r(x', y)/|\kappa(x')| < 1$ , so by Lemma 3.6 we get  $\partial r / \partial x < 0$ . Since  $L$  is  $C^{1,1}$ , the function  $r$  is Lipschitz (at least locally where it is finite), so its values near  $x$  can be computed by integrating this derivative. But this contradicts the fact that  $r$  is minimized at  $x$ .  $\square$

**Remark 3.12** In fact under the hypothesis of Lemma 3.11, the arc of  $L$  from  $x$  to  $y$  (in the direction of the tangent  $T$  at  $x$  with  $\langle T, y - x \rangle > 0$ ) must be an arc of a circle, but we will not need to invoke this stronger statement.

**Lemma 3.13** *Suppose  $\gamma$  is a subarc of  $L$  joining  $x$  to  $y$  with length at most  $\pi r(x, y)$ . Then  $\sup_{\gamma \cap E} |\kappa| \geq 1/r(x, y)$ , so  $\inf_{\gamma} \rho \leq r(x, y)$ .*

*Proof* In the case  $r(x, y) = \infty$  there is nothing to prove. Otherwise, for convenience we rescale so that  $r(x, y) = 1$  and translate so that  $C(x, y)$  is centered at the origin. Letting  $B$  denote the open unit ball,  $C(x, y)$  is then a great circle on  $\partial B$ .

First suppose there is a subarc  $\alpha \subset \gamma$  disjoint from  $B$  and with endpoints  $a, b \in \partial B$ . Then  $\alpha$  has length at most  $\pi$  but at least that of the great circle arc from  $a$  to  $b$ . Let  $\beta$  denote the extension of this latter arc (within the same great circle) with one endpoint at  $a$  and having the same length as  $\alpha$ . Since this is still less than a semicircle, the distance between the endpoints of  $\beta$  is at least  $|a - b|$ . Applying Schur's comparison theorem to  $\alpha$  and  $\beta$ , we conclude that the curvature of  $\alpha$  is somewhere at least that of  $\beta$ , i.e., that  $\sup_{\alpha} |\kappa| \geq 1$  as desired. (In [Sul08], we show that the standard proof [Che67] of Schur's theorem for smooth curves actually applies to all  $W^{1, BV}$  curves, that is to all curves of finite total curvature. In particular, it applies to  $C^{1,1}$  curves, with the curvature comparison being between the measures  $|\kappa| ds$ .)

If there is no such subarc, then  $B \cap \gamma$  is dense in  $\gamma$ . In particular there is a sequence  $x_i \in \gamma \cap B$  with  $x_i \rightarrow x$ . It now suffices to show  $\lim_{y \rightarrow x} |\kappa(y)| \geq 1$ .

The function  $f(p) := |p|^2 - 1$  is  $C^{1,1}$  along  $L$  with  $f(x) = 0 = f'(x)$ . Since  $f(x_i) < 0$  there is some  $y_i$  between  $x$  and  $x_i$  with  $f'(y_i) < 0$ , and thus some  $z_i$  between  $x$  and  $y_i$  such that  $f''(z_i) < 0$ . In fact the set of such  $z_i$  has positive measure, so we may choose  $z_i \in E$ . Then by the chain rule,

$$f''(z_i) = 2(1 + \langle z_i, \kappa(z_i) \rangle) > 2(1 - |z_i| |\kappa(z_i)|),$$

so we find that  $|\kappa(z_i)| |z_i| > 1$ . Since  $|z_i| \rightarrow 1$ , we have  $\lim |\kappa| \geq 1$  as desired.  $\square$

### 3.3 Thickness and stiff ropes

We can now prepare for the application of Clarke's theorem by expressing the reach of  $L$  as the minimum of a family of functions parametrized by the disjoint union  $(L \times L) \sqcup \text{Osc } L$ :

**Proposition 3.14** *For any  $C^{1,1}$  curve  $L$ ,*

$$\text{reach}(L) = \min \left\{ \frac{1}{2} \min_{x, y \in L} \text{pd}^*(x, y), \min_L \rho \right\} = \min \left\{ \frac{1}{2} \min_{x, y \in L} \text{pd}^*(x, y), \min_{c \in \text{Osc } L} R(c) \right\}.$$

*Proof* The right-hand sides are equal and by Lemmas 3.9 and 3.10 they are at least  $\text{reach}(L)$ . It remains to prove that either  $2\text{reach}(L) = \text{pd}^*(x, y)$  for some  $x, y \in L$ , or  $\text{reach}(L) = R(c)$  for some  $c \in \text{Osc } L$ .

By Corollary 3.4, we can find a sequence  $(x_i, y_i)$  with  $r^*(x_i, y_i) \rightarrow \text{reach}(L)$ . By compactness, a subsequence converges to some pair  $(x, y)$ . We consider three cases.

First, if  $x \neq y$  and  $y \in N_x L$  then  $\psi^*(x, y) = 0$ . Therefore,  $\text{pd}^*(x, y) = 2r^*(x, y) = 2\text{reach}(L)$ .

Second, if  $x \neq y$  and  $y \notin N_x L$ , then by Lemma 3.11 we have  $\text{reach}(L) = \rho(x)$ , which is the radius of some circle in  $\text{Osc} L_x$  by compactness.

Third, if  $x = y$ , then for large  $i$  the subarc  $\gamma_i$  from  $x_i$  to  $y_i$  satisfies the length bound of Lemma 3.13. Applying the lemma, we find a point  $z_i \in \gamma_i \cap E$  with  $1/|\kappa(z_i)| \leq r(x_i, y_i) + 1/i$ . Since  $z_i \rightarrow x$  while  $r(x_i, y_i) \rightarrow \text{reach}(L)$ , we conclude as desired that  $\rho(x) \leq \text{reach}(L)$ .  $\square$

Proposition 3.14 permits us also to model *stiff* ropes, which cannot bend as much as the reach constraint permits.

**Definition 3.15** If  $L$  is a  $C^{1,1}$  curve and  $\sigma \geq 1/2$ , we define the  $\sigma$ -**thickness** of  $L$  as

$$\text{Thi}_\sigma(L) := \min \left\{ 2\text{reach}(L), \frac{1}{\sigma} \min_L \rho \right\}.$$

We note that a link with  $\text{Thi}_\sigma \geq 1$  cannot have an osculating circle with radius less than  $\sigma$ . We specify  $\sigma \geq 1/2$  because otherwise this formula would simply give twice the reach. (It is tempting to try to define a thickness for  $\sigma < 1/2$  by combining the curvature term with a minimum distance of critical pairs. But this is unphysical in the sense that it permits the thick rope to penetrate itself near points of large curvature; furthermore it is not amenable to our analytic formulation since it is not bounded by reach.)

The next result writes thickness as a “min-function”, which will let us apply Clarke’s theorem.

**Corollary 3.16** For any link  $L$  and any  $\sigma \geq 1/2$  we have

$$\begin{aligned} \text{Thi}_\sigma(L) &= \min \left\{ \min_{x,y \in L} \text{pd}^*(x, y), \frac{1}{\sigma} \min_L \rho \right\} \\ &= \min \left\{ \min_{x,y \in L} \text{pd}(x, y), \min_{\substack{x \in \partial L \\ y \in L}} \text{pd}^*(x, y), \frac{1}{\sigma} \min_L \rho \right\}. \end{aligned}$$

*Proof* The first equality follows immediately from Proposition 3.14. The second follows from the fact that  $\text{pd}^*(x, y) \leq \text{pd}(x, y)$  with equality unless  $x \in \partial L$ .  $\square$

Clearly for any  $\sigma$  we have  $\text{Thi}_\sigma(L) = \infty$  if and only if  $L$  is a connected straight arc, since this is true of  $\text{reach}(L)$ . From Lemma 3.9 and the definition of  $\sigma$ -thickness we immediately get:

**Corollary 3.17** Suppose  $0 < \text{Thi}_\sigma(L) < \infty$ . If  $x, y \in L$  satisfy  $\text{pd}^*(x, y) = \text{Thi}_\sigma(L)$  then  $\text{Thi}_\sigma(L) = 2\text{reach}(L)$ , so  $(x, y) \in \text{Crit}(L)$ .  $\square$

**Definition 3.18** We refer to pairs  $(x, y) \in \text{Crit}$  with  $\text{pd}^*(x, y) = \text{Thi}_\sigma(L)$  as **struts**; and to circles  $c \in \text{Osc} L$  such that  $R(c) = \sigma \text{Thi}_\sigma(L)$  as **kinks**. We denote the sets of struts and kinks by

$$\text{Strut} = \text{Strut}(L) \subset \text{Crit} \subset L \times L, \quad \text{Kink} = \text{Kink}(L) \subset \text{Osc} L \subset C_3.$$

Thus the  $\sigma$ -thickness of  $L$  is realized exactly at the struts and kinks.

Every kink is a circle of the same radius  $\sigma$ , indeed it is a point in  $C_3$  of the form  $(x, T(x), n/\sigma)$  with  $|n| = 1$ . Thus we identify it with  $(x, n)$ , and we can and will view  $\text{Kink}(L)$  as a subset of the unit normal bundle to  $L$ . But without additional smoothness assumptions on  $L$  it is hard to say anything about the possible structure of this kink set.

The  $\sigma$ -ropelength problem is to minimize length subject to the condition  $\text{Thi}_\sigma \geq 1$ . For a closed link  $L$ , we minimize over the usual link type  $[L]$ . When  $L$  includes arc components, we constrain each endpoint  $p \in \partial L$  to lie in an affine subspace  $H_p^0$  (of dimension 0, 1 or 2). Furthermore we allow for Neumann or first-order boundary constraints by specifying that the tangent vector  $T(p)$  at each endpoint stay in a linear subspace  $H_p^1$ ; we consider only the cases of fixed tangents ( $\dim H_p^1 = 1$ ) and free tangents ( $\dim H_p^1 = 3$ ). We define the **constrained link type**  $[L]$  (as in [CFK<sup>+</sup>06, Section 8]) by requiring that each endpoint  $p$  stay on  $H_p^0$ , with tangent  $T(p) \in H_p^1$ , during any isotopy. (Of course it would be easy to allow more general constraint manifolds but we will not need this for our examples.)

To prevent isotopy classes from being too large, we could also include obstacles for the curve, as in [CFK<sup>+</sup>06]. The resulting wall struts in the criticality theory work just as in the Gehring problem considered there. However, in the examples we have in mind (like the simple clasp) the obstacles are never active constraints, so the wall struts are not needed. Thus we leave this extension of the theory as a straightforward exercise for the reader.

**Definition 3.19** Suppose  $\text{Thi}_\sigma(L) \geq 1$ . We say that  $L$  is a ropelength minimizer constrained by  $\sigma$ -thickness (or, for short, a  **$\text{Thi}_\sigma$ -constrained minimizer**) in its (possibly constrained) link type  $[L]$  if it minimizes length among all curves in  $[L]$  with  $\text{Thi}_\sigma \geq 1$ . We say  $L$  is a **local minimizer** if it minimizes length among all curves with  $\text{Thi}_\sigma \geq 1$  in some  $C^1$ -neighborhood.

**Proposition 3.20** *The thickness  $\text{Thi}_\sigma$  is upper semicontinuous with respect to the  $C^1$  metric on the space of  $C^{1,1}$  curves  $L$ .*

*Proof* By definition,  $\text{Thi}_\sigma$  is the minimum of  $\text{reach}(L)$  and a scaled radius-of-curvature term. Federer has shown [Fed59, Theorem 4.13] that  $\text{reach}(L)$  is upper semicontinuous even with respect to the topology induced by Hausdorff distance.

Thus it only remains to check that  $\min_L \rho$  is semicontinuous with respect to  $C^1$  convergence of  $L$ . Since this is a local function, it suffices to consider a connected curve  $L$ . Suppose  $L_i$  are  $C^{1,1}$  curves converging to  $L$ . As we have noted earlier, we may assume that the convergent  $C^1$  maps  $\gamma_i: L \rightarrow L_i$  each have constant speed  $v_i$  (with  $v_i \rightarrow 1$  of course). Now by the lower semicontinuity of Lipschitz constants, we have

$$\begin{aligned} (\min_L \rho)^{-1} &= \sup_{x \in E} |\kappa(x)| = \text{Lip}(T) \leq \liminf \text{Lip}(\gamma_i) = \liminf v_i^2 \sup_{x \in E_i} |\kappa_i(x)| \\ &= \lim(v_i^2) \liminf_{L_i} (\min \rho_i)^{-1} = \liminf_{L_i} (\min \rho_i)^{-1} \end{aligned}$$

which yields the desired conclusion.  $\square$

We now prove the existence of thickness-constrained minimizers, under a mild technical hypothesis that prevents the length of any component from shrinking to zero. Since a circle component of thickness  $\text{Thi}_\sigma \geq 1$  necessarily has length at least  $\pi$ , we only have to worry here about arc components. An arc component with endpoints  $p$  and  $q$  clearly has length bounded away from 0 if the constraints  $H_p$  and  $H_q$  are disjoint.

**Corollary 3.21** *Suppose the constrained link type  $[L]$  contains at least one curve  $L$  with  $\text{Thi}_\sigma(L) \geq 1$ , and suppose that, in at least one length-minimizing sequence  $L_i$  of such curves, the length of each component stays bounded away from zero. Then there exists a  $\sigma$ -thickness constrained minimizer in  $[L]$ .*

*Proof* We may assume the  $L_i$  are parametrized at locally constant speed on a common domain (say  $L_1$ ). By Arzela–Ascoli we may extract a subsequence converging in  $C^1$  to a limit curve  $L_0$ . (If the link  $L$  is split, we assume without loss of generality that the various pieces stay within a common ball while they shrink.) Because the convergence is in  $C^1$ , we have  $\text{len}(L_i) \rightarrow \text{len}(L_0)$ , and by Proposition 3.20 we know  $\text{Thi}_\sigma(L_0) \geq \liminf \text{Thi}_\sigma(L_i) \geq 1$ . That the endpoints of  $L$  still satisfy the given constraints is clear. Finally, by  $C^1$  convergence,  $L_0$  is isotopic to all but finitely many of the  $L_i$  and in particular,  $L_0 \in [L]$ .  $\square$

#### 4 The general balance criterion

We give an analytic condition, Theorem 4.15, that is both necessary and sufficient for a general curve to be critical for  $\sigma$ -ropelength (subject to the ancillary condition of  $\text{Thi}_\sigma$ -regularity). The condition may be viewed as an equation of vector distributions on  $\mathbb{R}^3$ . The approach follows the one we used in [CFK<sup>+</sup>06]: using Clarke’s Theorem 4.1 we compute the derivative of the thickness of a curve  $L$  under a variation induced by a smooth vector field  $\xi$ ; then we apply the Kuhn–Tucker theorem.

##### 4.1 The derivative of thickness

Here we give a formula for the first variation of the  $\sigma$ -thickness of  $L$ , which will be key to the technical definition of criticality for length subject to thickness constraints. The proof is an application of a theorem of Clarke [Cla75] on the directional derivatives of a function  $g$  that may be expressed as the minimum of a  $C^1$ -compact family  $\{g_u\}$  of  $C^1$  functions. Essentially this theorem states that the directional derivative of  $g$  at a point  $x$  is the minimum of the directional derivatives of those  $g_u$  for which  $g_u(x) = g(x)$ . In our case, this will mean that the first variation of thickness in the direction of a deforming vector field is given (in Theorem 4.5) as the minimum of the derivatives of the strut lengths and kink radii.

We use Clarke’s theorem in the following special case:

**Theorem 4.1 (Clarke)** *Let  $U$  be a sequentially compact topological space. Suppose that for each  $u \in U$  and some  $\varepsilon > 0$  there is a  $C^1$  function  $g_u: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  such that*

the functions  $(t, u) \mapsto g_u(t)$  and  $(t, u) \mapsto g'_u(t)$  are continuous. Then, putting  $g(t) := \min_{u \in U} g_u(t)$ , the right derivative of  $g$  exists at every  $t_0 \in (-\varepsilon, \varepsilon)$  and is given by

$$\left. \frac{dg}{dt^+} \right|_{t=t_0} = \min \{ g'_u(t_0) : u \in U, g_u(t_0) = g(t_0) \}.$$

□

That the *minima* exist (in the definition of  $g$  and the formula for its derivative) as opposed to *infima*, is of course an immediate consequence of the compactness hypothesis.

We have previously expressed thickness as the minimum of penalized distances between pairs of points on our curve and scaled radii over the closure of the set of osculating circles to  $L$ . It will be easy to differentiate penalized distances as we vary our curve, but somewhat more complicated to differentiate radii of curvature. We now turn to the task of defining and computing these derivatives.

While the main technical difficulties we face in this work are due to the fact that our curves may fail to be  $C^2$ , when we consider derivatives, it suffices to consider variations arising from smooth deformations of the ambient space  $\mathbb{R}^3$ : our balance criteria show that criticality just with respect to such variations suffices to get balancing measures.

We start by noting that any  $C^2$  diffeomorphism  $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  induces a diffeomorphism  $\phi_*$  on the space  $C_3$  of pointed circles:  $\phi_*(x, C)$  is the osculating circle at  $\phi(x)$  to the  $C^2$ -smooth curve  $\phi(C)$ . Working in the coordinates  $(x, C) = (x, T, \kappa)$  of Section 3.2, it is clear that  $\phi$  maps this circle  $C$  to a curve with velocity  $v := D_x \phi(T)$  and acceleration  $a := D_x^2 \phi(T, T) + D_x \phi(\kappa)$ . Thus

$$\phi_*(x, T, \kappa) = \left( \phi(x), \frac{v}{|v|}, \frac{a}{|v|^2} - \frac{\langle a, v \rangle v}{|v|^4} \right).$$

Expressing the length of the new curvature vector in the usual way in terms of the vector cross product gives

$$R(\phi_*(x, T, \kappa)) = \frac{|v|^3}{|v \times a|} = \frac{|D_x \phi(T)|^3}{|D_x \phi(T) \times (D_x^2 \phi(T, T) + D_x \phi(\kappa))|}.$$

Now consider a  $C^1$ -smooth family of  $C^2$  diffeomorphisms  $\phi^t$  with  $\phi^0 = \text{Id}$ . The initial velocity  $\left. \frac{d\phi^t}{dt} \right|_{t=0}$  will be some  $C^2$  vector field  $\xi$ . We get a  $C^1$ -smooth family  $\phi_*^t$  of  $C^2$  diffeomorphisms of  $C_3$ , whose initial velocity is a  $C^2$  vector field  $\xi_*$  on  $C_3$  depending only on  $\xi$ . The formula we need expresses the derivative of the radius function  $R$  in the direction  $\xi_*$  in terms of the given vector field  $\xi$  and its spatial derivatives.

**Lemma 4.2** *Given a  $C^1$ -smooth one-parameter family of  $C^2$  diffeomorphisms  $\phi^t$  with initial velocity  $\xi$ , the time derivative of the radius function  $R$  (where this is finite) is*

$$\delta_\xi R(x, T, \kappa) := D_{(x, T, \kappa)} R(\xi_*) = 2R \langle T, D_x \xi(T) \rangle - R^3 \langle \kappa, D_x^2 \xi(T, T) + D_x \xi(\kappa) \rangle.$$

*Proof* By smoothness, the time derivatives commute with spatial derivatives. From  $\phi^0 = \text{Id}$  we see  $D_x \phi^0 = \text{Id}$  and  $D_x^2 \phi^0 = 0$ . Thus we can write  $\delta_\xi R(x, T, \kappa)$  as

$$\begin{aligned} & \frac{3\langle T, D_x \xi(T) \rangle}{|T \times \kappa|} - \frac{\langle T \times \kappa, D_x \xi(T) \times \kappa + T \times (D_x^2 \xi(T, T) + D_x \xi(\kappa)) \rangle}{|T \times \kappa|^3} \\ &= 3R\langle T, D_x \xi(T) \rangle - R^3 \left( \langle T, D_x \xi(T) \rangle \langle \kappa, \kappa \rangle + \langle \kappa, D_x^2 \xi(T, T) + D_x \xi(\kappa) \rangle \right), \end{aligned}$$

using the facts that  $|T| = 1$  and  $|T \times \kappa| = 1/R$ . Since  $\langle \kappa, \kappa \rangle = R^{-2}$ , this reduces to the formula given.  $\square$

Of course if  $(x, T, \kappa)$  is the osculating circle to  $L$  at a point  $x \in E$ , then the quantity  $D_x^2 \xi(T, T) + D_x \xi(\kappa)$  appearing here is simply the second derivative  $\xi''$  of  $\xi$  along  $L$ .

**Corollary 4.3** *Suppose  $L$  is a  $C^{1,1}$  curve and  $\xi$  a  $C^2$  vector field on space. At any point  $x \in E_L$  with osculating circle  $c = (x, T, \kappa)$ ,  $\kappa \neq 0$ , we have*

$$\delta_\xi R(c) = 2R\langle \xi', T \rangle - R^3 \langle \xi'', \kappa \rangle.$$

**Lemma 4.4** *Suppose  $L$  is a  $C^{1,1}$  curve and  $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a  $C^2$  diffeomorphism. Then  $\phi_*(\overline{\text{Osc } L}) = \overline{\text{Osc } \phi L}$ .*

*Proof* By Lemma 2.1 we know  $\phi E_L = E_{\phi L}$ . Thus  $\phi_*(\text{Osc } L) = \text{Osc } \phi L$ ; since  $\phi_*$  is a homeomorphism, it follows that  $\phi_*(\overline{\text{Osc } L}) = \overline{\text{Osc } \phi L}$ .  $\square$

We are now ready to apply Clarke's theorem.

**Theorem 4.5** *Let  $\phi^t$  for  $t \in (-\varepsilon, \varepsilon)$  be a  $C^1$ -smooth family of  $C^2$  diffeomorphisms of  $\mathbb{R}^3$  with  $\phi^0 = \text{Id}$ , and let  $\xi$  be the initial velocity vector field*

$$\xi_x := \left. \frac{\partial \phi^t(x)}{\partial t} \right|_{t=0}.$$

*Let  $L$  be a  $C^{1,1}$  curve with  $\text{reach}(L) < \infty$ . Then the function  $t \mapsto \text{Thi}_\sigma(\phi^t(L))$  is differentiable from the right at  $t = 0$ , with right-hand derivative*

$$\begin{aligned} \delta_\xi \text{Thi}_\sigma(L) &:= \left. \frac{d \text{Thi}_\sigma(\phi^t(L))}{dt^+} \right|_{t=0} \\ &= \min \left( \min_{(x,y) \in \text{Strut}(L)} \frac{1}{2} \left\langle \frac{x-y}{|x-y|}, \xi_x - \xi_y \right\rangle, \frac{1}{\sigma} \min_{c \in \text{Kink}(L)} \delta_\xi R(c) \right). \end{aligned}$$

*Proof* We will apply Clarke's Theorem 4.1 to a family of functions of  $t$  parametrized by the compact space  $L \times L \sqcup \overline{\text{Osc } L}$ . The functions are the following: for  $(x, y) \in L \times L$  we use  $t \mapsto \text{pd}^{\phi^t(L)}(\phi^t(x), \phi^t(y))$ , and for  $c \in \overline{\text{Osc } L}$  we use  $t \mapsto 1/\sigma R(\phi^t(c))$ . These functions are  $C^1$  and both they and their derivatives depend continuously on the parameters, so Clarke's theorem applies.

By the lemma,  $\phi_*^t(\overline{\text{Osc } L}) = \overline{\text{Osc } \phi^t L}$ . Thus by Proposition 3.14 and the definition of  $\text{Thi}_\sigma$ , the minimum of our Clarke family is the thickness  $\text{Thi}_\sigma(\phi^t(L))$ . Clarke's



Theorem thus shows that thickness has a forward time derivative given by the minimum derivative of  $\text{pd}(x, y)$  or  $R/\sigma$  where these functions equal thickness.

By Corollary 3.17, struts are critical pairs: we have  $\text{pd}(x, y) = \text{Thi}_\sigma(L)$  only if  $(x, y) \in \text{Crit}$ . Differentiating the formula defining  $\text{pd}(x, y)$ , using the fact that  $\psi(x, y) = 0$ , we see that the derivative equals the derivative of  $|x - y|/2$  given above.  $\square$

Since superlinear functions may be characterized as infima of families of linear functions, we immediately get:

**Corollary 4.6** *Suppose  $L$  is a  $C^{1,1}$  curve with  $\text{reach}(L) < \infty$ . Then the operator  $\xi \mapsto \delta_\xi \text{Thi}_\sigma(L)$  is superlinear. That is, for  $a \geq 0$  and vector fields  $\xi$  and  $\eta$ , we have*

$$\delta_{a\xi} \text{Thi}_\sigma(L) = a\delta_\xi \text{Thi}_\sigma(L), \quad \delta_{\xi+\eta} \text{Thi}_\sigma(L) \geq \delta_\xi \text{Thi}_\sigma(L) + \delta_\eta \text{Thi}_\sigma(L).$$

#### 4.2 The balance criterion

Having computed the derivative of the function  $\text{Thi}_\sigma$  representing the one-sided constraint, we can now start to formulate our balance criterion. Recall that in a constrained link type, each endpoint  $p \in \partial L$  is constrained to lie in a subspace  $H_p$ .

**Definition 4.7** Let  $L$  be a  $C^{1,1}$  curve in the constrained link type  $[L]$ . Suppose  $\eta$  is a  $C^1$  vector field along  $L$  (for instance the restriction of  $C^2$  field on  $\mathbb{R}^3$ ). We say  $\eta$  is **compatible** with  $[L]$  at  $L$  if at each endpoint  $p \in \partial L$  we have that  $\eta$  is tangent to  $H_p^0$  and that  $\eta'(p) = D_p \eta(T) \in H_p^1$ . Assuming  $\text{reach}(L) < \infty$ , we say that  $L$  is  **$\text{Thi}_\sigma$ -regular** if it has a **thickening field**, meaning a compatible  $C^2$  vector field on  $\mathbb{R}^3$  with  $\delta_\eta \text{Thi}_\sigma(L) > 0$ .

Regularity is a form of constraint qualification; we will use it for instance to show that minimizers are critical points. Note that for a classical link type (with all components closed curves), any  $L$  with  $\text{Thi}_\sigma > 0$  is  $\text{Thi}_\sigma$ -regular: the Euler vector field  $\eta_p := p$  generating homotheties is a thickening field. Regularity is also easy to check for many examples of constrained links.

**Definition 4.8** Suppose  $\text{Thi}_\sigma(L) = 1$ . We say  $L$  is  **$\sigma$ -critical** if

$$\delta_\xi \text{len}(L) < 0 \implies \delta_\xi \text{Thi}_\sigma(L) < 0$$

for every compatible  $\xi \in C^2(L, \mathbb{R}^3)$ . We say  $L$  is **strongly  $\sigma$ -critical** if there exists  $\varepsilon > 0$  such that

$$\delta_\xi \text{len}(L) = -1 \implies \delta_\xi \text{Thi}_\sigma(L) \leq -\varepsilon$$

for every compatible  $\xi \in C^2(L, \mathbb{R}^3)$ .

Clearly strong criticality implies criticality. Under the assumption of  $\text{Thi}_\sigma$ -regularity they are in fact equivalent.

**Lemma 4.9** *If  $L$  is  $\text{Thi}_\sigma$ -regular and  $\sigma$ -critical, then  $L$  is in fact strongly  $\sigma$ -critical.*

*Proof* Let  $\eta$  be a thickening field for  $L$ . Scaling  $\eta$  if necessary, we may assume that  $\delta_\eta \text{len}(L) \leq 1/2$ . Thus for  $\xi$  as in the definition of strong criticality,  $\delta_{\xi+\eta} \text{len}(L) \leq -1/2$ . Using the superlinearity of Corollary 4.6, and the criticality of  $L$ , we get

$$0 > \delta_{\xi+\eta} \text{Thi}_\sigma(L) \geq \delta_\xi \text{Thi}_\sigma + \delta_\eta \text{Thi}_\sigma.$$

Thus we may take  $\varepsilon := -\delta_\eta \text{Thi}_\sigma(L)$ .  $\square$

The next two lemmas characterize  $\text{Thi}_\sigma$ -constrained local minimizers  $L$ . In the trivial case when  $\text{Thi}_\sigma(L) > 1$ , the thickness constraint is not active; if  $\text{Thi}_\sigma(L) = 1$  and  $L$  is  $\text{Thi}_\sigma$ -regular, then it is critical.

**Lemma 4.10** *If  $L$  is a  $\text{Thi}_\sigma$ -constrained local minimizer with  $\text{Thi}_\sigma(L) > 1$ , then each component of  $L$  is a straight arc.*

*Proof* Since the constraint  $\text{Thi}_\sigma \geq 1$  is not active at  $L$ , the curve is a local length minimizer without constraints. Thus  $\delta_\xi \text{len}(L) = 0$  for all compatible  $\xi$ , so  $L$  has zero curvature everywhere.  $\square$

**Lemma 4.11** *If  $L$  is a  $\text{Thi}_\sigma$ -constrained local minimizer with  $\text{Thi}_\sigma(L) = 1$ , and  $L$  is  $\text{Thi}_\sigma$ -regular, then  $L$  is (strongly)  $\sigma$ -critical.*

*Proof* Suppose  $\xi$  is a compatible vector field such that  $\delta_\xi \text{len}(L) < 0$ , but  $\delta_\xi \text{Thi}_\sigma \geq 0$ . Let  $\eta$  be a thickening field, and choose  $c > 0$  small enough that  $\delta_{\xi+c\eta} \text{len} < 0$ . By Corollary 4.6, we see  $\delta_{\xi+c\eta} \text{Thi}_\sigma > 0$ . Hence there are nearby curves in the same constrained link type with  $\text{Thi}_\sigma > 1$  but with smaller length, which is a contradiction.  $\square$

The rest of our results deal with strongly  $\sigma$ -critical curves  $L$  with  $\text{Thi}_\sigma(L) = 1$ , and thus apply to all  $\text{Thi}_\sigma$ -regular local minimizers (ignoring the trivial case of minimizers with  $\text{Thi}_\sigma(L) > 1$ , classified above). Our main theorem, the General Balance Criterion, says that a link is strongly critical if and only if it is balanced by certain measures on the kinks and struts.

**Definition 4.12** Let  $L$  be a  $C^{1,1}$  link. A **kink measure** for  $L$  is a nonnegative Radon measure on  $\text{Kink}(L)$ . A **strut measure** for  $L$  is a nonnegative Radon measure on  $\text{Strut}(L) \subset L \times L$  that is invariant under the interchange map  $(x, y) \mapsto (y, x)$ . Given a strut measure  $\mu$  on  $\text{Strut}(L)$  we define the **associated strut force measure**  $\Omega$  on  $L$  to be the vector-valued measure obtained by projecting the vector-valued Radon measure  $2(x - y)\mu(x, y)$  to  $L$  via  $(x, y) \mapsto x$ . Thus

$$\int_{\text{Strut}(L)} \langle x - y, \xi_x - \xi_y \rangle d\mu(x, y) = \int_L \langle \xi, d\Omega \rangle.$$

Physically one should think of a strut measure as a system of compressions on the points of self-contact of the embedded tube around  $L$ , or alternatively on certain compression-bearing elements of length 1 connecting critical pairs of  $L$ . The strut force measure then gives the resultant force along  $L$  itself. The physical interpretation of the kink measure is more elusive in general.

**Definition 4.13** A  $C^{1,1}$  link  $L$  with  $\text{Thi}_\sigma(L) = 1$  is  **$\sigma$ -balanced** if there exist a strut measure  $\mu$  (with strut force measure  $\Omega$ ) and a kink measure  $\nu$  for  $L$  such that for any compatible vector field  $\xi$  we have

$$\delta_\xi \text{len}(L) = \int_L \langle \xi, d\Omega \rangle + \int_{\text{Kink}(L)} \delta_\xi R(c) d\nu(c).$$

We refer to this as the **balance equation**. Note that it may be viewed as an equation of distributions acting on vector fields  $\xi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . The kink term has distributional order 2 by Lemma 4.2, while the other terms have order 0: in particular the variation of length can be written as

$$\delta_\xi \text{len}(L) = \int_L \langle \xi', T \rangle ds = - \int_L \langle \xi, \kappa \rangle ds + \sum_{p \in \partial L} \langle \xi, \pm T \rangle,$$

pairing  $\xi$  with a vector-valued Radon measure which is absolutely continuous on the interior and has outward-pointing atoms at each endpoint.

The General Balance Criterion is an application of the following version of the Kuhn–Tucker theorem from linear programming, which we proved in [CFK<sup>+</sup>06] following ideas of [Lue69]. As usual  $C(Y)$  denotes the space of continuous functions on a space  $Y$ .

**Theorem 4.14** *Let  $X$  be any vector space and  $Y$  be a compact topological space. For any linear functional  $f$  on  $X$ , and any linear map  $A : X \rightarrow C(Y)$ , the following are equivalent:*

- (a) *There exists  $\varepsilon > 0$  such that for each  $\xi \in X$  with  $f(\xi) = -1$  there exists  $y \in Y$  with  $(A\xi)(y) \leq -\varepsilon$ .*
- (b) *There exists a nonnegative Radon measure  $\mu$  on  $Y$  such that  $f(\xi) = \int_Y A(\xi) d\mu$  for all  $\xi \in X$ .*

□

**Theorem 4.15 (General Balance Criterion)** *A link  $L$  with  $\text{Thi}_\sigma(L) = 1$  is strongly  $\sigma$ -critical if and only if it is  $\sigma$ -balanced.*

*Proof* We apply Theorem 4.14 with  $X$  being the space of compatible vector fields  $\xi$  and  $f$  the linear functional  $f(\xi) := \delta_\xi \text{len}(L)$ . The idea is to capture the derivative  $\delta_\xi \text{Thi}_\sigma(L)$  as the minimum value of a continuous function  $A(\xi)$ . Thus following Theorem 4.5 we take  $Y := \text{Strut} \sqcup \text{Kink}$  and define  $A : X \rightarrow C(Y)$  via

$$A(\xi) := \begin{cases} \frac{1}{2} \langle x - y, \xi_x - \xi_y \rangle, & (x, y) \in \text{Strut}, \\ \sigma^{-1} \delta_\xi R(c), & c \in \text{Kink}. \end{cases}$$

The conclusion of Theorem 4.14 is then exactly that  $L$  is strongly critical if and only if it is balanced. □

The special case of a critical knot with no kinks was analyzed by Schuricht and von der Mosel [SvdM04]. Of course in this case our balance criterion reduces to theirs, involving only the strut measure.

**Proposition 4.16** *Suppose  $L$  is a critical link for the Gehring problem of minimizing length subject to maintaining distance 1 between components. If for some  $\sigma \geq 1/2$  we have  $\text{Thi}_\sigma(L) \geq 1$ , then  $L$  is also  $\sigma$ -critical.*

*Proof* The main theorem of [CFK<sup>+</sup>06] gives a strut measure on the set of Gehring struts (connecting points at distance 1 on distinct components). Under the assumption that  $\text{Thi}_\sigma(L) \geq 1$ , these Gehring struts are also struts in our sense. Even if there are kinks or further struts (between points on a single component) the Gehring strut measure alone balances the link, so by the General Balance Criterion it is  $\sigma$ -critical.  $\square$

Consider for instance, the known minimizing links from [CKS02], where each component is a convex planar curve built from straight segments and arcs of unit circles. By Lemma 4.11 they are  $1/2$ -critical. The same strut measure that balances them for the Gehring problem [CFK<sup>+</sup>06] also shows they are  $\sigma$ -balanced for any  $\sigma \leq 1$ . (For  $\sigma = 1$  the curved sections are kinks and balance can be achieved in other ways as well.)

#### 4.3 Kink-free arcs with special strut patterns

The kink term in the General Balance Criterion is a bit arcane; in Section 5 we will give nicer versions under certain minimal smoothness assumptions. But of course the kink term is irrelevant along kink-free arcs (or even kinked arcs over which the kink measure vanishes), so we can apply the General Balance Criterion directly.

**Lemma 4.17** *Suppose  $L$  is  $\sigma$ -balanced and  $A$  is an open subarc over which the kink measure vanishes. Then along  $A$  the strut force measure is absolutely continuous, given by  $\Omega = -\kappa ds$ .*

*Proof* For any vector field  $\xi$  vanishing on  $L \setminus A$  the kink term in the balance equation vanishes, so we get

$$\int_A \langle \xi, d\Omega \rangle = \delta_\xi \text{len}(L) = \int_A \langle \xi', T \rangle ds.$$

Integrating by parts gives the desired result.  $\square$

As a first application, we can easily analyze “free” sections of a critical curve, with no struts or kinks. (This result was first discussed – in the case of a  $C^\infty$  knot – by Gonzalez and Maddocks [GM99a].)

**Proposition 4.18** *If  $L$  is  $\sigma$ -balanced and  $A$  is a subarc with zero strut force measure and zero kink measure, then  $A$  is a line segment.*

*Proof* By the lemma  $\kappa ds = -\Omega = 0$  along the subarc.  $\square$

We now consider the case of two subarcs in “one-to-one contact”.

**Proposition 4.19** *Let  $L$  be  $\sigma$ -balanced. Suppose  $A$  and  $B$  are two subarcs with zero kink measure and suppose they are in one-to-one contact, meaning there is a homeomorphism  $\phi: A \rightarrow B$  such that there is a strut from  $a$  to  $\phi(a)$  for each  $a \in A$  but no other struts touching  $A \cup B$ . Then  $A \cup B$  forms a section of a standard symmetric double helix of pitch angle at least  $\pi/4$  (or of a circle).*

*Remark 4.20* We could start with the weaker assumption of a (weakly) monotonic family of struts, where a single point  $a \in A$  might touch a whole subarc  $B' \subset B$  or vice versa. In fact this cannot happen, since  $B'$  is a subarc of the unit normal circle to  $A$  at  $a$ , so the tangent vector has nonzero change along  $B'$ ; this would imply an atom of strut force measure at  $a$  which is impossible since  $\Omega$  is absolutely continuous on a kink-free arc.

*Proof* Change the orientation on  $B$  if necessary to assume that  $\phi$  is orientation-preserving. Since the kink measure vanishes on  $A \cup B$ , the lemma applies, giving  $\Omega = -T'$ . For any subarc  $aa' \subset A$ , by the symmetry of  $\Omega$  we get

$$T(a) - T(a') = \Omega(aa') = -\Omega(\phi(aa')) = T(\phi(a')) - T(\phi(a)).$$

This means that

$$W := T(a) + T(\phi(a))$$

is a constant vector along  $A$ .

Now define the continuous vector field

$$N(a) := \phi(a) - a,$$

along  $A$ . Since struts have unit length and  $\phi(a) \in N_a L$ , this is a unit normal field. Since  $\Omega$  acts in the direction  $-N$  of the single strut, we deduce that  $T' = |\kappa|N$  almost everywhere. That is,  $N$  is the Frenet principal normal.

Reversing the roles of  $A$  and  $B$ , we see equally well that  $N(a) \perp T(\phi(a))$ . (Indeed the principal normal at  $\phi(a) \in B$  is  $-N(a)$ .) It follows that  $N(a) \perp W$ , which in turn implies that  $\langle W, T(a) \rangle$  is constant along  $A$ , since

$$\langle W, T \rangle' = \langle W, T' \rangle = |\kappa| \langle W, N \rangle = 0.$$

But from the definition of  $W$ , we have

$$\langle W, T(a) \rangle = 1 + \langle T(a), T(\phi(a)) \rangle = \langle W, T(\phi(a)) \rangle,$$

so  $\langle W, T \rangle$  is the same constant along  $B$ .

Consider now the degenerate case where  $W = 0$ , meaning  $T(\phi(a)) = -T(a)$ . The arcs  $A$  and  $B$  stay in the plane of  $T(a)$  and  $N(a)$ , and indeed are centrally symmetric around the midpoint of any strut. Since  $a$  and  $\phi(a)$  are always at unit distance, it follows that  $A$  and  $B$  are antipodal arcs of a circle of diameter 1, a degenerate double helix of pitch zero.

Clearly this case only arises when  $\sigma = 1/2$ . Since points near  $\phi(a)$  are at distance less than 1 from  $a$ , it follows that  $A$  and  $B$  belong to the same component of  $L$ . Furthermore, by Remark 3.12, this component is the full circle of diameter 1. Since

this circle is kinked, balance could alternatively be obtained through a kink measure instead of the strut measure.

For the general case  $W \neq 0$ , think of  $W$  as a vertical vector. Since  $N \perp W$ , each strut connects points at equal height. Since  $\langle W, T \rangle$  is the same constant along each curve, the homeomorphism  $\phi$  is actually an isometry. Consider now the midpoints  $M(a) := (a + \phi(a))/2$  of the struts. Since  $\phi$  is an isometry, differentiating gives  $M' = W/2$ , meaning these midpoints move at constant speed in direction  $W$ . Since  $T$  makes a constant angle with  $W$ , the strut vectors  $N(a)$  also rotate at constant speed in the plane perpendicular to  $W$ . The arcs  $A$  and  $B$ , given as  $M \mp N/2$ , thus form a symmetric double helix as claimed.

(In the degenerate case where  $|W| = 2$ , we have  $T(\phi(a)) = T(a) \equiv W/2$ . That is, both  $A$  and  $B$  are straight segments, giving a degenerate double helix of pitch  $\pi/2$ . The strut measure vanishes on the struts connecting  $A$  and  $B$ .)

Consider the squared distance function from a fixed point  $(-1/2, 0, 0) \in A$  to the other strand  $B = \{(\cos \theta, \sin \theta, k\theta)/2\}$ . Since its second derivative is  $(k^2 - \cos \theta)/2$ , we see that it is convex (with a single minimum at the claimed strut) for  $k \geq 1$ , that is for pitch angle at least  $\pi/4$ . For smaller pitch angle, the distance has a local maximum at  $\theta = 0$ , so the thickness of the double helix is less than 1 and the curves are not in one-to-one contact.  $\square$

This agrees with the result of Maddocks and Keller [MK87] which states (under different hypotheses) that two intertwined ropes in equilibrium with one-to-one contact should form a double helix where the radii of the helices depend on the tension in the ropes. Schuricht and von der Mosel [SvdM04] show in this situation that the curvature vectors of  $A$  and  $B$  must point along the common strut, without carrying the analysis through to prove that the curves form a double helix.

## 5 Balance with regulated kinks

The General Balance Criterion can be hard to apply without some control on the kink set. In the balance equation, as we have already noted, the second-order kink term is equated to a distribution of order zero in the variation vector field  $\xi$ . Along a  $C^2$  subarc, there is of course at most one kink over each point of  $L$  and furthermore, Corollary 4.3 says the kink term can be expressed in terms of the second arclength derivative of  $\xi$ . In this case, standard distributional calculus then says this second-order term can be integrated by parts. Our goal is to carry this out even for less smooth links, like those in our examples. Over a junction point along a piecewise  $C^2$  curve, for instance, there may be two kinks. Our first theorem below says that we can essentially ignore such points: the kink measure is nonatomic even after projection down to  $L$ , so even any countable subset of  $L$  can be ignored.

In the later parts of this section we discuss the balance criterion under certain mild regularity assumptions about the kinked arcs of  $L$ ; these suffice first to guarantee a single kink over all but a countable subset of  $L$ , then to transfer the balance equations to distributions along  $L$ , and thus to apply the calculus of distributions. We end up with friendlier versions of the Balance Criterion, and can bootstrap to greater smoothness of the critical link  $L$ .

### 5.1 The projection of the kink measure is nonatomic

The kink measure  $\nu$  for a balanced link  $L$  is supported on  $\text{Kink}(L)$ , which we view as a subset of the unit normal bundle  $N_1(L)$  via  $(x, n) \longleftrightarrow (x, T(x), n/\sigma)$ . Thus we think of  $\nu$  as a measure on this circle bundle with support on  $\text{Kink}$ . We recall the projection  $\Pi: C_3 \rightarrow \mathbb{R}^3$ , in particular  $\Pi: N_1(L) \rightarrow L$ . If  $\nu$  is a kink measure for  $L$ , then we write  $\bar{\nu}$  for the projection of  $\sigma\nu$  to  $L$ , which of course is supported on  $\Pi\text{Kink}(L)$ . (The factor of  $\sigma$  here simplifies several formulas later.)

Using Lemma 4.2 we can write the kink term in the balance equation as

$$\begin{aligned} \int_{\text{Kink}} \delta_\xi R(x, n) d\nu(x, n) \\ = 2 \int_L \langle \xi', T \rangle d\bar{\nu}(x) - \sigma^2 \int_{\text{Kink}} \langle D_x^2 \xi(T, T), n \rangle d\nu(x, n) \\ - \sigma \int_{\text{Kink}} \langle D_x \xi(n), n \rangle d\nu(x, n). \end{aligned}$$

We note the linear and quadratic dependence on  $n$  in the last two terms; these could also be written as integrals over  $L$ , now with respect to projected vector- and tensor-valued measures. Thus it is really only the projections to  $L$  of the three measures  $\nu$ ,  $n\nu$  and  $(n \otimes n)\nu$  which enter into the balance equation. (What this essentially means is that if we Fourier-decompose the measure  $\nu$  on each normal circle, then it is only the components of order 0, 1 and 2 which matter.)

Our first result shows that no single normal circle has positive mass. This will later allow us to ignore countably many points along  $L$ .

**Theorem 5.1** *If  $L$  is  $\sigma$ -balanced, then the projection  $\bar{\nu}$  of the kink measure  $\nu$  to  $L$  is nonatomic.*

*Proof* Fix a point on  $L$ , which by translation we assume is at the origin. We must show that  $\nu(\Pi^{-1}\{0\}) = 0$ . We will obtain this equation as the limit of the balance equation applied to a family of variation fields  $\xi^\varepsilon$ .

Let  $\phi$  denote a smooth nonnegative bump function supported on the unit ball, with  $\phi \equiv 1$  in a small neighborhood of 0. Given any vector  $v \in \mathbb{R}^3$  we write  $v^\perp := v - \langle v, T_0 \rangle T_0$  for its part perpendicular to the tangent vector  $T_0 := T(0)$  at the origin. Then we define

$$\xi(x) = \xi^\varepsilon(x) := \phi(x/\varepsilon) x^\perp.$$

Since  $\xi^\varepsilon$  is supported on the  $\varepsilon$ -ball its  $L^\infty$  norm is  $O(\varepsilon)$ . Thus in the limit  $\varepsilon \rightarrow 0$  the order 0 (strut and  $\delta$  len) terms in the balance equation approach 0 (even though the strut force measure might have an atom at the origin). Therefore the kink term approaches 0 as well.

We easily calculate the derivatives

$$\begin{aligned} D_x \xi(v) &= D_{x/\varepsilon} \phi(v) x^\perp / \varepsilon + \phi(x/\varepsilon) v^\perp, \\ D_x^2 \xi(v, v) &= 2D_{x/\varepsilon} \phi(v) v^\perp / \varepsilon + D_{x/\varepsilon}^2 \phi(v, v) x^\perp / \varepsilon^2. \end{aligned}$$

Note that  $D\xi$  is  $O(1)$  while  $D^2\xi$  is  $O(1/\varepsilon)$ . At the origin (independent of  $\varepsilon$ ) we have  $D_0\xi(v) = v^\perp$ , while the second derivatives vanish.

Note that  $\xi^\varepsilon$  is supported on the  $\varepsilon$ -ball; since  $\text{reach}(L) \geq \text{Thi}_\sigma(L) = 1$  we know (from [?, Lemma 3.1]) that for small  $\varepsilon$  this ball contains a single arc  $\alpha^\varepsilon$  of  $L$  whose length is at most  $2\arcsin \varepsilon$ . Now suppose  $x \in \alpha^\varepsilon$  is at arclength  $s = O(\varepsilon)$  from 0. Then from the curvature bound we have  $|T(x) - T_0| \leq s/\sigma \leq 2|s|$  and thus  $|x - sT_0| \leq s^2$ . In particular,  $|T^\perp| = O(\varepsilon)$  and  $|x^\perp| = O(\varepsilon^2)$  along the whole arc  $\alpha^\varepsilon$ .

The integrand in the kink term is

$$\delta_{\xi^\varepsilon} R(x, n) = 2\sigma \langle T, D_x \xi^\varepsilon(T) \rangle - \sigma \langle n, D_x \xi^\varepsilon(n) \rangle - \sigma^2 \langle n, D_x^2 \xi^\varepsilon(T, T) \rangle.$$

First we show that this integrand is uniformly bounded as  $\varepsilon \rightarrow 0$ . Clearly the first two terms are  $O(1)$ . Writing

$$\langle n, D_x^2 \xi^\varepsilon(T, T) \rangle = 2D_{x/\varepsilon} \phi(T) \langle n, T^\perp \rangle / \varepsilon + D_{x/\varepsilon}^2 \phi(T, T) \langle n, x^\perp \rangle / \varepsilon^2$$

shows – using our estimates on  $T^\perp$  and  $x^\perp$  – that the third term is also  $O(1)$ . We also note that at  $x = 0$  the integrand reduces to

$$\delta_{\xi^\varepsilon} R(0, n) = 0 - \sigma \langle n, n \rangle - 0 = -\sigma.$$

Now as  $\varepsilon \rightarrow 0$  the arcs  $\alpha^\varepsilon$  shrink to the single point  $\{0\}$ , so since the kink integrand is uniformly bounded, the kink integral  $\int_{\Pi^{-1}(\alpha^\varepsilon)} \delta_{\xi^\varepsilon} R(x, n) d\nu$  approaches

$$\int_{\Pi^{-1}\{0\}} \delta_{\xi^\varepsilon} R(x, n) d\nu = -\sigma \nu(\Pi^{-1}\{0\}).$$

Thus this measure is zero, as desired.  $\square$

## 5.2 Regularly balanced links

**Definition 5.2** Suppose a link  $L$  is  $\sigma$ -balanced by strut measure  $\mu$  and kink measure  $\nu$ . We say  $L$  is **regularly balanced** if there is an open subset  $U \subset L$  such that  $\bar{\nu}(L \setminus U) = 0$  and the unit tangent  $T$  has regulated derivative  $\kappa$  on  $U$ .

We conjecture that every balanced link is regularly balanced, but this seems difficult to prove. But there is a condition on  $L$  which will ensure this. We say a  $C^{1,1}$  curve  $L$  has **regulated kinks** if  $\Pi \text{Kink}$  is contained in a submanifold  $M \subset L$  on which  $T$  has regulated derivative.

**Lemma 5.3** Suppose  $L$  has regulated kinks. Then  $L$  is regularly balanced if and only if  $L$  is balanced (if and only if  $L$  is strongly  $\sigma$ -critical).

*Proof* It only remains to show that if  $L$  is balanced then it is regularly balanced. We set  $U := M \setminus \partial M$ . We know  $\bar{\nu}$  is supported on  $\Pi \text{Kink} \subset M$ . Since  $\partial M$  is countable and  $\bar{\nu}$  is nonatomic, we have  $\bar{\nu}(L \setminus U) = 0$ .  $\square$



In the rest of this section we analyze regularly balanced links to get several equivalent conditions that are easier to apply. First we show that we can reformulate the balance equation to involve distributions along  $L$  instead of on  $\mathbb{R}^3$ ; then we integrate by parts twice, ending with a balance equation that can be stated as an equality of measures with no explicit variation vector field. This is the condition we use later to show our examples are (regularly) balanced.

Suppose  $L$  is regularly balanced. We let  $J$  denote the jump set of  $\kappa$  on  $U$ ; since  $J$  is countable and  $\bar{\nu}$  is nonatomic,  $\bar{\nu}(J) = 0$ . Over each point of  $U \setminus J$  there is at most one kink; a kink exists only when  $|\kappa| = 1/\sigma$ . (Over each point in  $J$  there are at most two kinks; but we may ignore these with regards to the kink measure.) By Lemma 2.3 we may replace  $U$  by an open subset on which  $|\kappa| \geq 1/2\sigma$ . Thus the unit principal normal vector  $N := \kappa/|\kappa|$  is well-defined as a regulated function on  $M$  (with jumps only on  $J$ ).

**Lemma 5.4** *On a regularly balanced link  $L$ , the kink measure  $\nu$  is uniquely determined by its projection  $\bar{\nu}$ , and the kink term in the balance equation becomes*

$$\int_{\text{Kink}} \delta_{\xi} R(x, n) d\nu(x, n) = \int_U (2\langle \xi', T \rangle - \sigma \langle \xi'', N \rangle) d\bar{\nu},$$

using Corollary 4.3. □

Here we note that in the last term, both  $N$  and  $\xi''$  are regulated functions (with jumps only on  $J$ ). Since their product is also regulated and  $\bar{\nu}$  is nonatomic, the integral is well-defined.

By this lemma, the balance equation for a regularly balanced  $L$  can be expressed entirely in terms of derivatives of the vector field  $\xi$  along the curve  $L$ . Of course,  $\xi$  here is still a  $C^2$  vector field in space, and the balance equation is an equation of distributions on such vector fields. Our next result shows, however, that we can translate it into an equation of distributions on  $C^2$  vector fields along  $L$ . (We recall that the  $C^2$  structure on  $L$  comes not directly from the embedding in  $\mathbb{R}^3$  but instead from the local identification with  $\mathbb{R}$  given by an arclength parametrization.) This sets us up to use the standard calculus of distributions: by examining the highest-order term, we can integrate by parts and bootstrap to higher smoothness.

**Theorem 5.5** *Let  $L$  be a link with  $\text{Thi}_{\sigma}(L) = 1$ . Then  $L$  is regularly balanced by strut force measure  $\Omega$  and kink measure  $\nu$  if and only if*

$$\int_L \langle \eta', T \rangle ds - \int_L \langle \eta, d\Omega \rangle = \int_U (2\langle \eta', T \rangle - \sigma \langle \eta'', N \rangle) d\bar{\nu}$$

for all compatible  $C^2$  vector fields  $\eta \in C^2(L, \mathbb{R}^3)$  along  $L$ .

Note that this is the same balance equation we already have for  $C^2$  fields on space – the only difference is that it is now supposed to hold for  $C^2$  fields along  $L$ .

*Proof* First suppose this balance equation holds for all compatible  $\eta \in C^2(L, \mathbb{R}^3)$ . Given a  $C^2$  vector field  $\xi$  on space, to check the balance equation for  $\xi$  it suffices to find a sequence of compatible smooth fields  $\eta_i$  along  $L$  with uniformly bounded

$C^2$  norms such that  $|\eta_i - \xi|_{C^1(L)} \rightarrow 0$  and  $\eta_i'' \rightarrow \xi$  pointwise on  $U \setminus J$ . For then each term in the balance equation for  $\eta_i$  approaches the corresponding term for  $\xi$  (in Lemma 5.4). In particular, to handle the second-order term  $\int_{U \setminus J} \langle N, \eta_i'' \rangle d\bar{v}$  we use the dominated convergence theorem. But the construction of the  $\eta_i$  is easy: we simply start with the restriction of  $\xi$  to  $L$  and smooth it by convolving with a sequence of mollifiers. (Small modifications near the endpoints suffice to maintain the compatibility conditions.) Since  $\xi''$  is regulated on  $U$  with jumps only on  $J$ , the desired pointwise convergence follows from Lemma 2.2.

Conversely, if  $L$  is regularly balanced, then given any compatible  $C^2$  field  $\eta$  along  $L$  it suffices to find a sequence of smooth  $\xi_i$  on  $\mathbb{R}^3$  that have uniformly bounded  $C^2$  norms, that converge to  $\eta$  in  $C^1(L)$  and whose second derivatives converge pointwise on  $U \setminus J$ . Indeed it suffices to construct the  $\xi_i$  locally in a neighborhood of any given point  $p \in L$ ; these pieces can be patched together with a partition of unity. By translation we assume  $p = 0$  and let  $T_0$  be the tangent there. The idea is to extend  $\eta$  to  $\bar{\eta}$  on a neighborhood of  $0 \in \mathbb{R}^3$  by making  $\bar{\eta}$  constant on each plane perpendicular to  $T_0$ , and then smooth this in space.

More precisely, consider the function  $f: x \mapsto \langle T_0, x \rangle$ . Restricted to  $L$ , it is  $C^{1,1}$  and has regulated second derivative on  $U$ . On some neighborhood  $V \subset L$  of  $p$  we have  $1/2 < f' \leq 1$ , so in particular  $f|_V$  is a  $C^1$  diffeomorphism onto its image  $(a, b) \subset \mathbb{R}$ . Lemma 2.4 applies to show the inverse function  $g: (a, b) \rightarrow V$  is a  $C^{1,1}$  parametrization with speed in  $[1, 2)$ , and has regulated second derivative on the subset  $f(U \cap V)$ . Thus if we set  $\bar{\eta} := \eta \circ g$  then  $\bar{\eta}$  is also  $C^{1,1}$  with regulated second derivative on  $f(U \cap V)$ . To get the  $\xi_i$ , we simply smooth  $\bar{\eta}$  by convolving it with a sequence of mollifiers:

$$\xi_i := (\bar{\eta} * \phi_i) \circ f.$$

The desired properties again follow immediately using Lemma 2.2.  $\square$

On a regularly balanced link  $L$ , we have discussed the principal normal  $N$  as a regulated function on  $U$ . For convenience we extend it arbitrarily outside of  $U$ . (Of course for points  $x \in E$  with  $\kappa \neq 0$  we are free to pick  $N = \kappa/|\kappa|$  but this will be irrelevant.) The vector-valued measure  $N\bar{v}$  vanishes outside  $U$  since  $L$  does. In the balance equation of Theorem 5.5, we can thus equally well write the integral over  $U$  as an integral over all of  $L$ .

**Lemma 5.6** *Suppose  $L$  is regularly balanced. Then the projected kink measure  $\bar{v}$  is absolutely continuous with respect to  $ds$  and indeed there exists  $\Phi \in W^{1,\text{BV}}(L, \mathbb{R}^3)$  such that  $N\bar{v} = \Phi ds$  and  $\Phi(p) \perp H_p^1$  at each endpoint  $p \in \partial L$ . The balance equation for  $L$  can then be written as*

$$\int_L \langle \eta, d\Omega \rangle = \int_L \langle \eta', T - 2|\Phi|T - \sigma\Phi' \rangle ds.$$

*Proof* The balance equation from Theorem 5.5 equates  $\int_L \langle \eta'', N d\bar{v} \rangle$  with terms of order at most one in  $\eta$ , so this term is also order one. Thus we can write  $N\bar{v} = \Phi ds$  with  $\Phi \in \text{BV}(L, \mathbb{R}^3)$ . Since  $\bar{v}$  is nonnegative, it follows that  $\Phi = |\Phi|N$ ; of course

$|\Phi| \in \text{BV}(L)$  is nonnegative and vanishes (a.e.) outside  $U$ . Now we may integrate by parts to obtain

$$-\int_L \langle \eta'', N \rangle d\bar{v} = -\int_L \langle \eta'', \Phi \rangle ds = \int_L \langle \eta', \Phi' \rangle ds - \sum_{p \in \partial L} \langle \pm \eta', \Phi \rangle$$

where  $\pm \eta'$  is the derivative of  $\eta$  in the outward direction  $\pm T$ . Note that the value  $\Phi(p)$  of a BV function at an endpoint is well-defined as the one-sided limit.

Thus we may write the balance equation from Theorem 5.5 as

$$\int_L \langle \eta', T \rangle ds - \int_L \langle \eta, d\Omega \rangle = \int_L \langle \eta', 2|\Phi|T + \sigma \Phi' \rangle ds - \sigma \sum_{p \in \partial L} \langle \pm \eta', \Phi \rangle.$$

Since the left-hand side has order 0, so does the right-hand side. Our first conclusion is that the atomic terms  $\langle \eta', \Phi \rangle$  vanish at each endpoint. Since a compatible vector field  $\eta$  can have an arbitrary value  $\eta'(p) \in H_p^1$  at  $p \in \partial L$ , this simply means that  $\Phi(p) \perp H_p^1$ . The balance equation then reduces to the form given in the lemma.

Our second conclusion is that the integrand  $2|\Phi|T + \sigma \Phi'$  (which gets paired with  $\eta'$ ) is a BV function. Since  $T$  and  $|\Phi|$  are both BV, so is their product and we conclude that  $\Phi' \in \text{BV}$ , i.e., that  $\Phi \in W^{1,\text{BV}}(L, \mathbb{R}^3)$ , as desired. In particular  $\Phi$  is continuous.  $\square$

A few comments on the boundary conditions are in order. Let  $p \in \partial L$  be an endpoint. By continuity it is clear that  $\Phi(p)$  is a normal vector. Thus if  $\dim H_p^1 = 1$  (that is, if the tangent vector at  $p$  is fixed) then the condition  $\Phi \perp H_p^1$  is automatic. If on the other hand  $\dim H_p^1 = 3$  (that is, if the tangent vector is free) then of course  $\Phi \perp H_p^1$  means  $\Phi(p) = 0$ .

**Corollary 5.7** *If  $L$  is regularly balanced then the vector field  $\Phi$  of Lemma 5.6 vanishes on the jump set  $J \subset U$  of  $\kappa$ .*

*Proof* Suppose  $x \in J$  is a jump point of  $\kappa$ . If at least one one-sided limit has  $|\kappa|(x\pm) < 1/\sigma$ , then there are no kinks in some one-sided neighborhood of  $x$ . Thus  $\bar{v}$  vanishes on that neighborhood and so does  $\Phi$ , so  $\Phi(p) = 0$  by continuity. Otherwise, the jump in  $\kappa$  reflects a jump between kinks in different normal directions, that is,  $N$  also has a jump at  $x$ . But the continuity of  $\Phi$  implies that  $N = \Phi/|\Phi|$  is continuous at any point where  $\Phi \neq 0$ . Thus again we conclude  $\Phi(p) = 0$ .  $\square$

**Definition 5.8** Suppose  $L$  has  $\text{Thi}_\sigma = 1$ . A **kink tension function** for  $L$  is a nonnegative  $\phi \in W^{1,\text{BV}}(L)$ , vanishing at any endpoint  $p \in \partial L$  with free tangent vector, such that on the open set  $U := \{p \in L : \phi(p) > 0\}$  the link  $L$  is  $C^2$  with constant curvature  $|\kappa| \equiv 1/\sigma$ . We call the BV vectorfield

$$V := (1 - 2\phi)T - \sigma(\phi N)'$$

the **virtual tangent** associated to  $\phi$ , noting that it agrees with  $T$  outside  $U$ .

We are now ready to give our final reformulation of the balance criterion.

**Definition 5.9** Suppose  $L$  has  $\text{Thi}_\sigma = 1$ . We say  $L$  is **nicely balanced** if it has a strut measure  $\mu$  (with strut force measure  $\Omega$ ) and a kink tension function  $\phi$  (with virtual tangent  $V$ ) such that  $\Omega + V' = 0$  as measures on the interior of  $L$ , while at each endpoint  $p \in \partial L$ , we have  $\Omega\{p\} \mp V(p) \perp H_p^0$ .

Note that this nice form  $\Omega = -V'$  of the balance equation generalizes the equation  $\Omega = -T'$  for kink-free arcs (where of course  $V = T$ ) from Lemma 4.17. Our second main result now follows.

**Theorem 5.10** *A link  $L$  is regularly balanced if and only if it is nicely balanced.*

*Proof* Suppose first that  $L$  is regularly balanced. In view of Lemma 5.6 we set  $\phi := |\Phi|$ . Since this is continuous,  $\{\phi > 0\}$  is open, and we may replace the original  $U$  (in the definition of regularly balanced) by this open subset. Since  $\phi$  vanishes on  $J$  by Corollary 5.7, we know that  $L$  is  $C^2$  on  $U$ . In terms of the virtual tangent  $V = (1 - 2\phi)T - \sigma\Phi'$ , the balance equation of the lemma is  $\int_L \langle \eta, d\Omega \rangle = \int_L \langle \eta', V \rangle ds$ . Integrating by parts gives  $\Omega + V' = 0$  on the interior and  $\langle \eta, \Omega\{p\} \mp V(p) \rangle$  at each endpoint  $p \in \partial L$ . Recalling that a compatible vector field  $\eta$  can have any value parallel to  $H_p^0$  at  $p$ , we obtain  $\Omega\{p\} \mp V(p) \perp H_p^0$ .

Conversely, if  $L$  is nicely balanced with strut measure  $\mu$  and kink tension function  $\phi$ , we define  $\bar{v} := \phi ds$ . Since  $L$  is  $C^2$  along  $U = \{\phi > 0\}$  there is a unique kink measure  $\nu$  projecting to this  $\bar{v}$ . Retracing our steps in the integrations by parts, we see that  $L$  is regularly balanced by  $\mu$  and this  $\nu$ .  $\square$

We summarize our main results as

$$\begin{aligned} \text{nicely balanced} &\iff \text{regularly balanced} \\ &\implies \text{balanced} \iff \text{strongly critical} \implies \text{critical}. \end{aligned}$$

We also have the following partial converses: a balanced link with regulated kinks is nicely balanced; a critical link that is  $\text{Thi}_\sigma$ -regular is strongly critical. We recall that every closed link – with only circle components – is regular. We conclude for instance that a closed link with regulated kinks is  $\sigma$ -critical if and only if it is nicely balanced.

We note that it would be possible to do the analysis of this section for a single subarc  $A \subset L$ . If  $A$  has regulated kinks, then the kink measure over  $A$  can be expressed in terms of a kink tension function and virtual tangent. If  $A$  abuts other kinked arcs, the boundary conditions of course get more complicated. We have not carried this out in detail even though it would allow a slight strengthening of the results below on strut-free kinked arcs – we would only need to assume regulated kinks along the arc in question rather than on the whole link.

Given Theorem 5.10, we can rephrase the conjecture mentioned above as follows:

**Conjecture 5.11** Every balanced link is nicely balanced. In particular, the kink measure is supported over piecewise  $C^2$  arcs of the link.

We gain some hope that this conjecture is true from the analysis above: we have seen, for instance, that if an arc  $A$  has regulated kinks but the jump set  $J$  of  $\kappa$  is dense in  $A$ , then the kink measure vanishes over  $A$ . The effect of the kink measure, as seen in the kink tension function, grows only in the interior of  $C^2$  pieces of the link.

**Corollary 5.12** *Suppose  $L$  is nicely balanced with kink tension  $\phi$ . Then along  $U$  we have  $L \in W_{\text{loc}}^{3,\text{BV}}(U, \mathbb{R}^3)$ . The normal  $N$  and thus also the binormal  $B := T \times N$  are in  $W_{\text{loc}}^{1,\text{BV}}(U)$ , so the torsion  $\tau := \langle N', B \rangle$  is locally BV on  $U$ .*

*Proof* Recall that  $\phi \in W^{1,\text{BV}}(L)$  and  $\phi > 0$  on  $U$ . Since  $(1/\phi)' = -\phi'/\phi^2$  we see that  $1/\phi \in W_{\text{loc}}^{1,\text{BV}}(U)$ . Since  $\phi N = \Phi \in W^{1,\text{BV}}(L, \mathbb{R}^3)$  we conclude that  $N \in W_{\text{loc}}^{1,\text{BV}}(L, \mathbb{R}^3)$ . But on  $U$ , we have  $N = \sigma \kappa$ , so this means  $L \in W_{\text{loc}}^{3,\text{BV}}(U, \mathbb{R}^3)$ , as claimed. From the product rules, we see  $B := T \times N \in W_{\text{loc}}^{1,\text{BV}}(U, \mathbb{R}^3)$  and then  $\tau := \langle N', B \rangle \in \text{BV}_{\text{loc}}(U)$ .  $\square$

It follows that along  $U$  we have the usual Frenet equations

$$T' = N/\sigma, \quad N' = -T/\sigma + \tau B, \quad B' = -\tau N.$$

We can thus write

$$\begin{aligned} V &= (1 - \phi)T - \sigma\phi'N - \sigma\tau\phi B, \\ V' &= ((1 - \phi)/\sigma - \sigma\phi'' + \sigma\tau^2\phi)N - \sigma(\tau'\phi + 2\phi'\tau)B. \end{aligned}$$

Along  $U$  we may decompose the restricted strut force measure  $\Omega|_U$  into two signed Radon measures

$$\Omega|_U = \Omega_N N + \Omega_B B, \quad \Omega_N := \langle \Omega, N \rangle, \quad \Omega_B := \langle \Omega, B \rangle.$$

We now rewrite the balance equation  $\Omega = V'$  in terms of these measures.

**Corollary 5.13** *If  $L$  is nicely balanced, then we have the following equalities of signed Radon measures on  $U$ :*

$$\begin{aligned} \sigma^2\phi'' + (1 - \sigma^2\tau^2)\phi &= 1 + \sigma\Omega_N, \\ \sigma(\phi^2\tau)' &= \phi\Omega_B. \end{aligned}$$

**Corollary 5.14** *At a point  $p \in U$ , an atom of  $\Omega_N$  corresponds to a jump in  $\phi$ , while an atom of  $\Omega_B$  corresponds to a jump in  $\tau$ . If  $\Omega\{p\} = 0$  at a limit point  $p$  of  $L \setminus U$ , then  $\phi'(p) = 0$ . If  $\Omega\{p\} = 0$  at an isolated point  $p$  of  $L \setminus U$ , then  $\phi'_+(p) + \phi'_-(p) = 0$  and if these are nonzero then  $N$  changes sign at  $p$ .*

*Proof* From the equation  $\Omega = -V'$  and the fact that  $(1 - 2\phi)T$  is continuous, we see that

$$\text{atom of } \Omega \longleftrightarrow \text{jump in } V \longleftrightarrow \text{jump in } (\phi N)'.$$

Thus on  $U$ , an atom of  $\Omega_N$  corresponds to a jump in  $\phi'$  while an atom of  $\Omega_B$  corresponds to a jump in  $\phi^2\tau$ , i.e., to a jump in  $\tau$ .

Now recall that  $\phi \equiv 0$  on  $L \setminus U$ . Thus if  $p$  is a limit point, at least one of the one-sided derivatives  $\phi'_\pm(p)$  vanishes. If  $\Omega$  has no atom at  $p$ , the derivative  $\phi'(p)$  exists, hence is 0.

Finally, suppose  $p$  is an isolated point of  $L \setminus U$ . If  $\Omega$  has no atom there, then  $\phi'N$  is continuous at  $p$ , which yields the desired conclusion.  $\square$

## 6 Strut-free arcs

We want to consider arcs of critical curves that are balanced by kink measure alone. In the absence of strut force, it is convenient to ignore struts completely and to rescale such that kinks have curvature 1. Essentially, we take a limit of the constraints  $\sigma \text{Thi}_\sigma(L) \geq 1$  as  $\sigma \rightarrow \infty$ , and are left with the curvature constraint

$$\text{Thi}_\infty(L) := \min_L \rho \geq 1.$$

It should be clear that the derivative of  $\text{Thi}_\infty$  is like that of  $\text{Thi}_\sigma$  but sees only the kink terms, and that our General Balance Theorem adapts to this situation to say  $L$  is **strongly  $\infty$ -critical** if and only if it is **balanced by kink measure alone**. In case  $L$  has regulated kinks, it is of course regularly and indeed nicely balanced as before.

**Lemma 6.1** *Suppose  $L$  is  $\sigma$ -balanced, and  $A$  is a compact subcurve such that the strut force measure  $\Omega$  vanishes along the interior of  $A$ . (In particular this is the case if there are no struts with endpoints in the interior of  $A$ .) Then the rescaled curve  $A/\sigma$  has  $\text{Thi}_\infty \geq 1$ . Considered as a curve with fixed endpoints and fixed tangent directions there,  $A/\sigma$  is balanced by kink measure alone, and is thus strongly  $\infty$ -critical. Conversely, if  $A$  is strongly  $\infty$ -critical, then for any  $\sigma \geq 1/\text{reach}(A)$  we find that  $\sigma A$  is  $\sigma$ -balanced.*

*Proof* For the first direction, note that even if some struts to  $A$  carry strut measure necessary to balance other parts of the curve, they have by assumption no net effect on  $A$  and thus can be ignored when balancing  $A$ . The endpoint constraints on  $A$  ensure there is no restriction on the kink measure there.

For the converse, note first that  $\text{Thi}_\sigma(\sigma A) \geq 1$ . In the case  $\sigma = 1/\text{reach}(A)$ , the curve  $\sigma A$  may have some struts, but even then it can be balanced with  $\mu = 0$ .  $\square$

**Remark 6.2** For this problem of minimizing length subject only to the curvature constraint  $\text{Thi}_\infty \geq 1$ , we can treat each component of a link separately. As in Figure 2 (right), the curves do not necessarily stay embedded: we may have nonembedded critical configurations. Thus we should generalize our setup to allow nonembedded  $C^{1,1}$  curves.

In this section, we classify connected strongly  $\infty$ -critical curves under the assumption that they have regulated kinks. That is, we classify connected curves which are nicely balanced by kink measure alone. Of course each such curve has positive reach if it is embedded, and is thus  $\sigma$ -critical for large enough  $\sigma$ , but we do not compute the reach for our individual examples. By the lemma above, any strut-free arc of a  $\sigma$ -balanced link will be one of the types in our list.

To get started, suppose  $L$  is a connected curve, nicely balanced by kink measure alone. Since  $V' = -\Omega = 0$ , the virtual tangent

$$V = (1 - 2\phi)T - (\phi N)'$$

is constant along  $L$ . Since we have rescaled to get  $\sigma = 1$ , the equations from Corollary 5.13 for the kink tension function  $\phi$  along  $U := \{\phi > 0\}$  become

$$\phi'' + (1 - \tau^2)\phi = 1, \quad (\phi^2 \tau)' = 0. \quad (1)$$

Thus on each component  $W \subset U$  we see that  $\phi^2\tau$  is some constant  $c$ . On  $W$  we can then express (1) as the semilinear ODE

$$\phi'' = 1 - \phi + \frac{c^2}{\phi^3} \quad (2)$$

for  $\phi$  and we can write

$$V = (1 - \phi)T + \phi'N + \frac{c}{\phi}B. \quad (3)$$

Then along  $W$  we have

$$|V|^2 = (\phi - 1)^2 + \phi'^2 + \frac{c^2}{\phi^2}$$

and this of course is a conserved quantity for the ODE. In the  $(\phi, \phi')$  phase plane (or for  $c \neq 0$ , in the  $\phi > 0$  halfplane) this is clearly a proper, strictly convex function. Thus it has a single minimum at some  $(\phi_c, 0)$  and its other level sets are closed loops encircling this minimum. It follows that all solutions to (2) are periodic.

**Proposition 6.3** *Suppose a connected curve  $L$  is nicely balanced by kink measure alone and suppose at some point  $p \in L$  we have  $\phi^2\tau \neq 0$ . Then  $\phi^2\tau = c$  is constant – and in particular  $\phi > 0$  satisfies (2) – along all of  $L$ . The kink tension function  $\phi$  on such  $L$  is uniquely determined.*

*Proof* As above, let  $W$  be the component of  $\{\phi > 0\}$  containing  $p$ . On  $W$  we know  $\phi$  satisfies (2). Since  $c \neq 0$  the level set of  $|V|^2$  is a closed loop in the halfplane  $\phi > 0$ , meaning the solution extends with nonvanishing  $\phi$  to the whole curve  $L$ . For the final statement, note first that  $\phi$  is uniquely determined up to a constant factor by  $\phi^2\tau = c$ ; the constant is then determined by (2).  $\square$

**Lemma 6.4** *Suppose an arc from  $p$  to  $q$  is nicely balanced by kink measure alone, with  $\phi > 0$  and virtual tangent  $V$ . Then*

$$\langle q - p, V \rangle = \phi'(q) - \phi'(p) - c^2 \int_p^q \phi^{-3} ds.$$

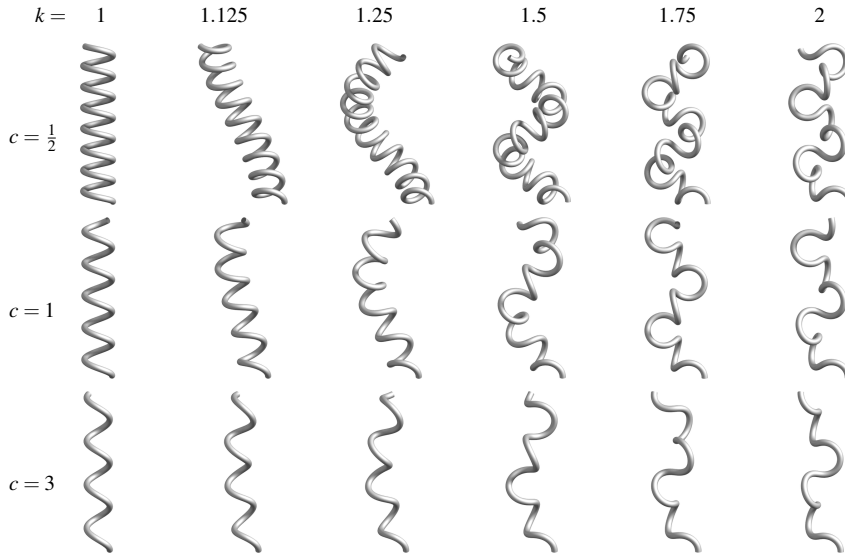
*Proof* From (3) and (2) we have

$$\langle q - p, V \rangle = \int_p^q \langle T, V \rangle ds = \int_p^q (1 - \phi) ds = \int_p^q \phi'' ds - c^2 \int_p^q \phi^{-3} ds.$$

**Corollary 6.5** *For  $c \neq 0$  no solution to (2) gives a closed curve.*

*Proof* Any solution  $\phi$  is periodic along  $L$ , but for  $c \neq 0$  the lemma means that each period of the curve makes the same negative progress  $-c^2 \int_L \phi^{-3} ds$  in the direction of the virtual tangent  $V$ . Thus we cannot close up after any number of periods.  $\square$

Even if the general solutions to (2) cannot be expressed in closed form, it is easy to integrate the ODE numerically for different values of  $c$  and different initial conditions. Given their shapes (see Figure 1) we call these curves **supercoiled helices**. We conjecture that each supercoiled helix is embedded. We can restate Proposition 6.3 as



**Fig. 1** The picture shows  $\sigma$ -critical curves obtained by solving (2) with a variety of initial conditions and values for  $c$ . For any  $c$ , there is one solution with constant  $\phi \equiv \phi_c$ : a helix by Lemma 6.6. this is a helix. The various solutions shown have initial conditions  $\phi'(0) = 0$  and  $\phi(0) = k\phi_c$  for various multiples  $k$  of  $\phi_c$ . The initial value of  $\phi$  increases from left to right, while the value of  $c$  increases from top to bottom. The shape of the curves explains why we call them supercoiled helices; they become progressively more twisted as  $k$  increases. The virtual tangent  $V$  is vertical in all of these pictures.

follows: Suppose a connected curve  $L$  has nonzero torsion somewhere and is nicely balanced by kink measure alone. Then  $L$  is a subarc of some supercoiled helix.

This same family of curves was discovered by Hector Sussmann, who called them “helical arcs”. Sussmann gives a fascinating control-theoretic derivation of the family in his research abstract [Sus95]. He considers the same problem of minimizing length subject to the curvature bound  $\text{Thi}_\infty \geq 1$  for arcs with fixed endpoints and fixed tangents there. He shows the absolute length minimizer (for any given boundary conditions) is either a helical arc or a concatenation of at most three circular arcs and straight segments (as in our case  $c = 0$  below). Our results are somewhat weaker than Sussmann’s in that he has fewer regularity assumptions, but are stronger in that we classify all *critical* curves, rather than just minimizers. (Sussmann also claims to have a proof that any supercoiled helix is a local strict minimizer for length in the sense that each subarc of length less than some  $\delta > 0$  is the unique length minimizer for its endpoints, but the promised paper with details does not seem to have appeared even as a preprint.)

There is one case in which we can integrate the ODE (2) explicitly – when  $\phi \equiv \phi_c$  is constant:

**Lemma 6.6** *Suppose a connected curve  $L$  is nicely balanced by kink measure alone. If the torsion is nonzero somewhere and  $\phi \neq 0$  is constant along  $L$ , then  $L$  is a helix with constant torsion  $|\tau| < 1$ . The virtual tangent  $V$  points along the axis of the helix.*



*Proof* Since  $\phi > 0$  we know  $\kappa \equiv 1$ ; since  $\phi \equiv \phi_c$  is constant, equation (1) reduces to  $1 - \tau^2 = 1/\phi$ , showing that  $|\tau| < 1$  is constant and  $L$  is a helix. To check the last assertion, note that the tangent vector  $T$  makes a constant angle with the virtual tangent: from (3) we have  $\langle T, V \rangle = 1 - \phi_c < 0$ .  $\square$

*Remark 6.7* Suppose we consider the one-parameter family of helices

$$\gamma_r(t) := (r \cos t, r \sin t, t),$$

with curvature  $r/(1+r^2)$  and torsion  $1/(1+r^2)$ . The curve-shortening flow decreases  $r$  while staying in this family. Thus it increases curvature for  $r > 1$  (i.e. for  $|\tau| < \kappa$ ) but decreases curvature for  $r < 1$ . This immediately verifies that  $\gamma_r(t)$  cannot be critical for  $r < 1$ , while our result that it is critical for  $r > 1$  is reasonable.

Now we turn to the case  $c = 0$ . Based on what we have already proved about the case  $c \neq 0$ , we see that if  $c = 0$  on one component  $W$  of  $U \subset L$ , then we must have  $c = 0$  on all of  $U$ . Thus  $\tau \equiv 0$  on  $U$ , so each component of  $U$  is an arc of a unit circle (if not the whole circle). Thus  $L$  is made up of (potentially infinitely many) circular arcs (the components of  $U$ ) possibly joined by straight segments ( $L \setminus U$ ). We will use Corollary 5.14 to analyze the possible junctions.

First we examine the possible kink tension functions  $\phi$  on a circular arc, noting that for  $c = 0$  equations (2), (3) become

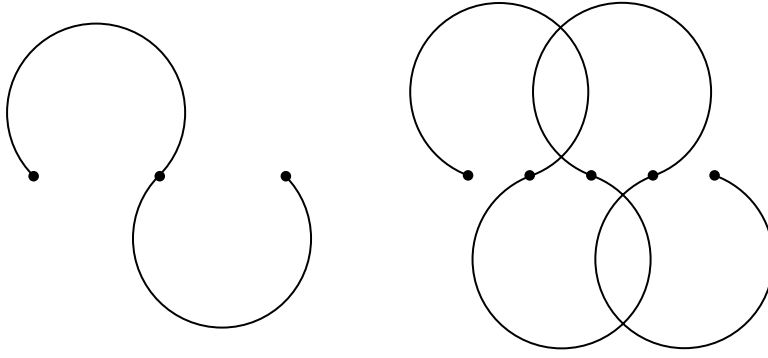
$$\phi'' = 1 - \phi, \quad V = (1 - \phi)T + \phi'N.$$

So suppose that  $L$  is a unit circle. Given any vector  $V$  in the plane of  $L$ , we define  $\phi := 1 - \langle T, V \rangle$  on  $L$ . Clearly  $\phi \geq 0$  on  $L$  if and only if  $|V| \leq 1$ . That is, the various possible kink measures balancing  $L$  correspond to the virtual tangent vectors  $V$  in the closed unit disk. For  $V = 0$  we have  $\phi \equiv 1$  (and it is interesting to think of  $L$  as a degenerate helix in the context of Lemma 6.6). For  $|V| < 1$  we have  $\phi > 0$  on  $L$ . For  $|V| = 1$  we have  $\phi > 0$  except at a single point  $p \in L$  where  $\phi(p) = 0 = \phi'(p)$ .

For  $|V| > 1$ , we cannot use this  $\phi$  to balance the whole circle, but we do have  $\phi > 0$  on an arc of more than half the circle, centered at the point where  $T = -V$ ; at its endpoints  $\phi = 0$  but  $\phi' \neq 0$ . Congruent such arcs can be joined end-to-end in a  $C^1$  fashion such that  $V$  remains constant at each junction point while  $N$  flips sign. (See Figure 2.) We call an infinite such concatenation a **wave**. A wave is embedded if and only if the turning angle of each piece is less than  $5\pi/3$ , that is, if and only if  $|V| > 2/\sqrt{3}$ . (Of course, waves with smaller  $V$  do have embedded subarcs.)

**Theorem 6.8** *Suppose  $L$  is an embedded connected curve, nicely balanced by kink force alone (for fixed endpoints with fixed tangents). If  $L$  has any point of nonzero torsion, then as we have seen, it is a subarc of some supercoiled helix (for instance a helix of torsion less than 1). Otherwise  $L$  is either a straight segment (possibly joined to circular arcs at each end), a circle (or arc thereof), or a subarc of some wave.*

*Proof* We have already treated the case of nonzero torsion, so we may assume  $c = 0$ . Thus the curve is made up of straight segments and unit circular arcs. At any junction between two pieces we have  $\phi = 0$ , and by Corollary 5.14 we have  $\phi' = 0$  unless  $N$  flips sign.



**Fig. 2** A wave is the planar  $C^1$  concatenation of circular arcs of the same turning angle  $\theta > \pi$ . On the left, we see such an example. Since the straight line joining these endpoints is also critical, this shows that there are many  $\sigma$ -critical curves joining the same pair of fixed endpoints. If we allow nonembedded curves, there are infinitely many such critical configurations, like the one on the right.

Our classification now proceeds according to  $|V|$ . Along any straight segment we have  $V = T$ , so  $|V| = 1$ ; if the segment is joined to a circular arc at either end, this  $V$  uniquely determines the kink tension function on that arc. In particular the embeddedness of  $L$  means each arc is less than a full circle, so we never have  $\phi = 0$  again along either arc and there are no further junctions.

If  $|V| < 1$  on a circular arc then  $\phi > 0$  so there are no junctions and  $L$  is a circle (or some subarc). (Here  $V$  is not uniquely determined. Since  $L$  is embedded we do not go more than once around the circle.)

Finally if  $|V| > 1$  on a circular arc, then if the arc extends to where  $\phi = 0$  we have  $\phi' \neq 0$  so if there is a junction it is exactly the kind seen in a wave. Extending, there can be further junctions, but the whole curve is a subarc of the wave specified by  $V$ . (If there is no junction, we are really in the previous case of a circular arc. As long as there is at least one junction,  $V$  is uniquely determined.)  $\square$

*Remark 6.9* If we did allow nonembedded curves, then there would be additional examples as follows: at any point  $p \in L$  where  $\phi = 0 = \phi'$  (for instance any point along a straight segment of  $L$ ), we can splice in a “hoop”, a full circle tangent to  $L$  at  $p$ . Indeed we could traverse many different hoops at  $p$  before continuing further along the initial curve  $L$ . Comparing where we used embeddedness in the proof above, we see these (along with circles traversed more than once) are the only new examples.

**Corollary 6.10** *Suppose  $L$  is an embedded connected curve, nicely balanced by kink force alone (for fixed endpoints with free tangents). Then  $L$  is either a straight segment, a circle, or the subarc of a wave between some two junction points (that is, a planar  $C^1$  concatenation of circular arcs with equal turning angle  $\theta > \pi$ ).*

*Proof* Since the tangent vectors at the endpoints are free, we must have  $\phi = 0$  there. That means we are looking for those examples from the theorem that satisfy this boundary condition. (Recall that on almost all examples,  $\phi$  was uniquely determined.) Supercoiled helices are clearly excluded. In the other three examples, the endpoints are restricted to the special cases listed.  $\square$

*Remark 6.11* Analogous to Remark 6.7 we can give the following intuition for the condition that each piece in a wave has turning angle greater than  $\pi$ . Consider the one-parameter family of circular arcs through two fixed points in a plane. The curvature is maximized at the semicircle. The arcs of less than a semicircle can thus be shortened while decreasing curvature – even staying within our family – while the arcs of more than a semicircle cannot.

Durumeric [?] used Sussmann's work to prove that every closed  $C^{1,1}$  curve which is a local minimum for ropelength has at least one strut. In our language, such curves are  $1/2$ -minimizing. We now prove a similar result which again is weaker in that it requires regulated kinks but stronger in that it applies to all critical curves, not just to minimizers.

**Corollary 6.12** *Every closed  $1/2$ -critical curve with regulated kinks has at least one strut.*

*Proof* If the curve has nonzero strut force measure, it must have struts. If not, the curve is a circle of unit diameter by Theorem 6.8, and it again has struts.  $\square$

It is also interesting to see how two arcs of the type we have been considering can join at a point  $p$  where there is an atom of strut force measure. At  $p$  the virtual tangent  $V$  jumps by exactly  $\Omega\{p\}$ , while of course  $\phi$  is continuous. If  $\phi(p) = 0$  we are talking about a junction between circular arcs (or perhaps one straight segment); here the atom of  $\Omega$  allows us to change the plane of the circle (and to change  $\phi'$ ).

If on the other hand  $\phi(p) > 0$ , the Frenet frame is well-defined, and we now consider atoms in  $\Omega_N$  and in  $\Omega_B$  separately using Corollary 5.14. At an atom of  $\Omega_B$  we have a jump in  $c = \phi^2\tau$  but  $\phi'$  (like  $\phi$ ) is continuous. That is, we might change from one supercoiled helix to another, or might jump to or from the case  $c = 0$ . At an atom of  $\Omega_N$ , on the other hand,  $c$  stays constant but  $\phi'$  jumps. For  $c \neq 0$  this means a horizontal jump in the phase space, generally to a different supercoiled helix with the same  $c$ . For  $c = 0$  we don't see any effect on the curve at  $p$  – it remains a circular arc – but the jump in  $\phi'$  affects where  $\phi$  vanishes to either side along this arc. As an example of this last case, we prove a lemma which will be needed later in our discussion of tight clasps.

**Lemma 6.13** *Suppose a kinked circular arc  $\gamma$  of turning angle  $2\theta$  is joined at each end to straight segments. Suppose further that this configuration bears no strut force except for a single atom at some point  $p$ . Then the configuration is balanced if and only if  $p$  is the midpoint of  $\gamma$  and the atom acts in the normal direction  $-N$  with mass  $2\sin\theta$ .*

*Proof* Let  $T_1$  and  $T_2$  be the tangent vectors to the straight segments. Since  $V = T$  on these segments, the jump in  $V$  is exactly

$$T_2 - T_1 = (2\sin\theta)N(q),$$

where  $N(q)$  is the normal vector at the midpoint  $q$  of  $\gamma$ . This jump must cancel the atom of strut force measure. Since the strut force always acts in the normal plane and  $N(q)$  is normal to the curve only at  $p$ , we see  $p = q$  as claimed.  $\square$

## 7 Noncompact curves

Sometimes it is interesting to consider noncompact (but still metrically complete) curves  $L$ . Since a complete curve  $L$  with positive reach is properly embedded, for any compact  $K \subset \mathbb{R}^3$ , the intersection  $L \cap K$  is compact. Typically (e.g., by Sard's theorem for almost every closed ball  $K$ ) this intersection is actually a compact subcurve of  $L$ .

Of course the length of  $L$  is infinite, but if we restrict our attention to variations  $\xi$  supported on some compact  $K \subset \mathbb{R}^3$  then  $\delta_\xi \text{len}(L)$  and  $\delta_\xi \text{Thi}_\sigma(L)$  are given by the same formulas as before, noting that only those struts and kinks touching  $K \cap L$  – a compact subfamily – matter here.

Fix now a compact  $K$  and a complete curve  $L$  with  $\text{Thi}_\sigma(L) = 1$ . We say that  $L$  is strongly  $\sigma$ -critical for variations supported on  $K$  if there exists  $\varepsilon > 0$  (possibly depending on  $K$ ) such that the condition in the earlier definition of strong criticality holds for all  $\xi$  supported on  $K$ . We say that  $L$  is  $\sigma$ -balanced for variations supported on  $K$  if there exist strut and kink measures (possibly depending on  $K$ ) such that the balance equation holds for all  $\xi$  supported on  $K$ .

It is straightforward to extend the General Balance Criterion (for each  $K$ ) to say that  $L$  is strongly critical for variations supported on  $K$  if and only if it is balanced for variations supported on  $K$ . Indeed, in the typical case when  $K \cap L$  is a compact subcurve  $A$ , this statement is only slightly different from the General Balance Criterion for  $A$  (considered with any new endpoints and their tangents fixed): Essentially the parts of  $L$  at distance at most 1 from  $K$  act as obstacles for  $A$ .

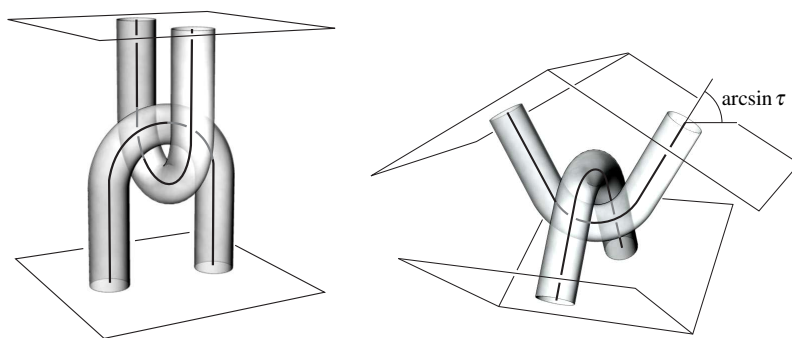
Now suppose for a complete curve  $L$  with  $\text{Thi}_\sigma(L) = 1$  we can find a single strut measure  $\mu$  and a single kink measure  $\nu$  (typically given by a kink tension function  $\phi \in W_{\text{loc}}^{1,\text{BV}}(L)$  vanishing outside  $C^2$  arcs) such that the balance equation holds for all compactly supported  $\xi$ . It follows for each  $K$  that  $L$  is strongly critical for variations supported on  $K$ . In particular,  $L$  is critical – any compactly supported variation that decreases length must also decrease thickness.

In previous sections, we have implicitly seen several examples like this already:

- A straight line is balanced by  $\mu = 0$  and  $\nu = 0$ .
- A infinite double helix of pitch angle at least  $\pi/4$  is balanced by the single family of struts in one-to-one contact.
- Any supercoiled helix is balanced by the  $\phi > 0$  used to define it; in particular any infinite single helix with  $\tau < |\kappa|$  is balanced by a constant  $\phi$ .
- Any infinite wave (with each piece having turning angle more than  $\pi$ ) is balanced by its  $\phi$ , which vanishes at every junction.

With appropriate regularity and smoothness assumptions, one can show these are the only complete critical curves with the kink/strut patterns we considered before, i.e., kink-free with controlled strut pattern as in Section 4.3, or strut-free as in Section 6.

In the clasps we discuss next, the ends of each arc – attached the boundary planes – are straight segments. Clearly we could extend these to be infinite rays and talk about a complete clasp. It would be balanced by the same compactly supported strut and kink measures used for the compact clasp.



**Fig. 3** The pictures show the “simple clasp” problem. On the left, we see the basic clasp where the endpoints are constrained to lie in parallel planes. On the right, we have the angled clasp where the four ends of the rope make an angle of  $\arcsin \tau$  with the horizontal. We will study  $\sigma$ -critical clasp configurations for varying values of  $\tau$  and  $\sigma$ .

## 8 The tight clasp

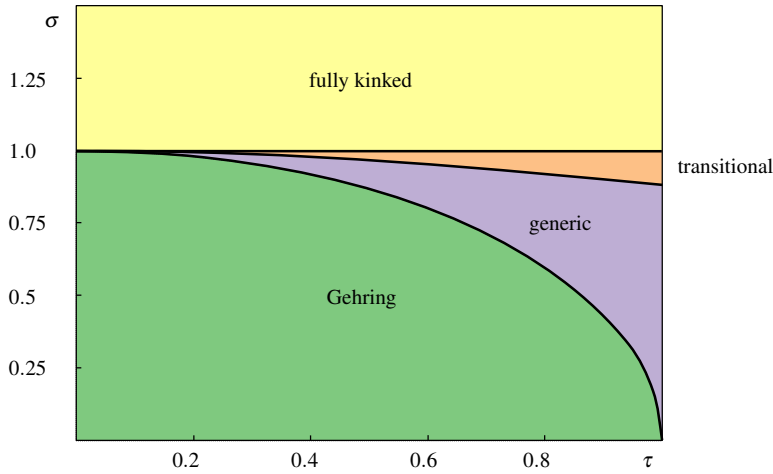
Our next example is a variation on the “simple clasp” which we considered previously in [CFK<sup>+</sup>06, Sect. 9]. There we studied the problem of minimizing the total length of a system  $L$  of two interlooped ropes, one anchored to the floor and one to the ceiling, subject to the condition that the two strands are everywhere separated by at least unit distance (see Figure 3).

In fact, we considered the entire family of “ $\tau$ -clasp” problems,  $0 \leq \tau \leq 1$ , in which the four ends of the two ropes are no longer vertical but make an angle of  $\arcsin \tau$  with the horizontal (thus the case  $\tau = 1$  is the basic clasp described above). In each case we described in detail a critical configuration (a “Gehring clasp”) that we conjectured to be minimizing. Here we consider the analogous problem in the more physically realistic setting of the present paper.<sup>1</sup>

**Definition 8.1** Suppose that the endpoints of two arcs are constrained to lie on the faces of a large tetrahedron with dihedral angles  $2\arcsin \tau \in [0, \pi]$  on two edges which form an orthogonal frame with the line connecting their midpoints, as in Figure 3. The  $(\tau, \sigma)$ -clasp problem is the problem of minimizing the length of this configuration subject to the constraint that  $\text{Thi}_\sigma(L) \geq 1$ , such that the two loops obtained by concatenating the segments joining the endpoints are linked. A **critical curve** for the  $(\tau, \sigma)$ -clasp problem is a  $\sigma$ -critical curve which obeys the constraints.

In this section we construct critical curves for the various  $(\tau, \sigma)$ -clasp problems. We believe these solutions to be minimizing, but we do not see how to prove it. The curves that we obtain fall into four regimes, depending on the values of the parameters  $\tau$  and  $\sigma$ , as shown in the phase diagram of Figure 4). In each case they consist of two congruent arcs lying in orthogonal planes. Both components are symmetric with

<sup>1</sup> It makes sense in this context to consider the modified Gehring problem in which, in addition to the unit separation of the two strands, we insist that the curvature of each strand never exceed  $1/\sigma$ . For this problem we permit the stiffness to assume the full range of values  $0 \leq \sigma < \infty$ . The resulting general theory



**Fig. 4** This phase diagram shows the domain of the various types of solutions to the clasp problem as the values of  $\tau$  (the arcsin of the angle made by the endpoints of the clasp with the horizontal) and  $\sigma$  (the stiffness parameter) change. In the uppermost “fully kinked” region, the clasp is a pair of circle arcs of radius  $\sigma$  joined with straight segments. There is a single strut connecting these arcs. In the next “transitional” region, the clasp consists of arcs of circles of radius  $\sigma$  at the tips joined by straight segments to arcs of circles of radius 1 at the shoulders of the clasp, finally joined by straight segments to the endpoints. In the third “generic” region, the curve is piecewise analytic, with eleven analytic pieces: a circle arc of radius  $\sigma$  at the tip, joined by straight segments to arcs of the “Gehring clasp” from [CFK<sup>+</sup>06]. These arcs are joined by straight segments to circle arcs of unit radius, which are joined by straight segments to the endpoints of the clasp. In the last, “Gehring” region, the solution is the same as that from [CFK<sup>+</sup>06].

respect to the line of intersection of the two planes, which we take to be the  $z$ -axis. We describe the component lying in the  $xz$ -plane, which we take to be the one with endpoints attached to the ceiling, as in [CFK<sup>+</sup>06]. In the discussion below, we will refer to a circular arc of maximal curvature  $1/\sigma$  as a *kink*.

- $\sigma \geq 1$ : the fully kinked regime. Here the curve consists of a kink of total angle  $2\arcsin \tau$ , with straight segments attached to the endpoints. There is exactly one strut between the two components, joining their tips (the points lying on the  $z$ -axis).
- $\frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}} \leq \sigma < 1$ : the transitional regime. In this case the curve consists of a kink of angle  $2\arcsin \frac{\tau}{2}$  joined by line segments to two circular arcs of radius 1 and angle  $\arcsin \tau - \arcsin \frac{\tau}{2}$ , each centered at the tip of the other component. There is a one parameter family of struts connecting each point of the latter arcs to tip of the other.
- $\sqrt{1-\tau^2} < \sigma < \frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}}$ : the generic regime. This is the most complicated possibility, of which the others may all be regarded as degenerations. The curve is piecewise analytic, with eleven analytic pieces, described by four parameters  $a, b, \alpha, \beta$  (determined in section 8.5 below): a kink of angle  $2\alpha$  at the tip; joined to

of critical curves follows the same lines as above and in [CFK<sup>+</sup>06], and we refrain from developing it explicitly here.

two straight segments of length  $a$ ; each joined to a section of the Gehring  $\tau$ -clasp described by the parameter interval  $[\arcsin \alpha, \arcsin \beta]$ ; each joined to another straight segment of length  $b$ ; joined to a circular arc of radius 1, centered at the tip of the other component, and of angle  $\arcsin \tau - \beta$ ; each joined finally to a straight segment connected to a constraining plane. There are two types of one-parameter families of struts connecting the two components: first, those connecting the arcs of radius 1 to the tip of the other component; second, each point of each Gehring arc shares a strut with the conjugate points (in the sense of [CFK<sup>+</sup>06]) of the two Gehring arcs of the other component.

- $0 \leq \sigma \leq \sqrt{1 - \tau^2}$ : the Gehring regime. For these parameter values the critical curves are identical to those described in [CFK<sup>+</sup>06].

### 8.1 General results on clasp-type curves

We start with some useful lemmas about configurations of circular arcs.

**Lemma 8.2** *Suppose a  $\sigma$ -critical link  $L$  passes through the origin and includes the circular arc  $C := \{(\sin \theta, 0, \cos \theta) : \theta_0 \leq \theta \leq \theta_1\}$ . If  $\sigma < 1$  so that  $C$  is not kinked and if  $C$  has no struts except those to the origin, then these struts generate an atom of strut force measure at the origin whose vertical component has magnitude  $\sin \theta_1 - \sin \theta_0$ .*

*Proof* Since  $C$  has no kinks,  $\Omega(C)$  is the difference in the tangent vectors at the two ends of  $C$ . This force all gets transmitted to the origin.  $\square$

**Lemma 8.3** *Let  $C$  be circle in the  $xz$ -plane, centered on the  $z$ -axis, and let  $B$  be a  $C^1$  arc in the  $yz$ -plane. If  $(p, q) \in B \times C$  is a critical point for the distance, and  $p$  is an interior point of  $B$ , then at least one of the points  $p, q$  lies on the  $z$ -axis.*

*Proof* Since  $(p, q)$  is critical for distance, the segment  $\overline{pq}$  is normal to  $B$  and  $C$ . Therefore, if  $q$  does not lie on the  $z$ -axis then the projection of  $p$  to the  $xz$ -plane must be the center of  $C$ . It follows that all points of  $C$  are equidistant from  $p$ . However, unless  $p$  itself lies on the  $z$ -axis then not all of the segments  $\overline{pr}$  joining  $p$  to  $r \in C$  are normal to  $B$  at  $p$ , contradicting the criticality of the pair  $(p, r)$ .  $\square$

We describe configurations of the clasp where the two components are congruent plane curves, lying in planes perpendicular to each other. To fix their symmetries in coordinates, let one component lie in the  $xz$ -plane while the other lies in  $yz$ -plane.

**Definition 8.4** Consider the point symmetry group of order eight in  $O(3)$ , algebraically isomorphic to  $D_4$ , and consisting of mirror reflections across the  $xz$ - and  $yz$ -planes, together with a four-fold rotary reflection around the  $z$ -axis. Using the Conway-Thurston orbifold notation, we call this group  $2 * 2$ .

In each of our descriptions of a clasp, we will describe only the portion of the clasp in a fundamental domain for this symmetry. This will be a convex curve in the halfplane of the  $xz$ -plane with positive  $x$ ; its endpoint on the  $z$ -axis will be called the **tip** of the clasp. It will sometimes be convenient for us to parametrize this curve by the sine  $u$  of the angle that its tangent makes with the  $x$ -axis.

We will be interested in proving that the minimum distance between two such arcs is at least 1. To this end we adapt Lemma 9.3 of [CFK<sup>+</sup>06].

**Lemma 8.5** *Let  $\gamma_1$  and  $\gamma_2$  be two convex curves lying in the  $xz$ - and  $yz$ -planes respectively. Suppose there is a critical pair  $(p_1, p_2)$  of length  $\rho$  connecting these components. Write  $x_i$  for the distance from  $p_i$  to the  $z$ -axis, and  $u_i$  for the sine of the angle between the tangent to  $\gamma_i$  and the horizontal. Then  $0 \leq x_i \leq u_i \leq 1$ , and any two of the numbers  $x_1, x_2, u_1, u_2$  determine the other two according to the formulas*

$$x_i^2 = \rho^2 - \frac{x_j^2}{u_j^2} = \rho^2 \frac{u_i^2(1 - u_j^2)}{1 - u_i^2 u_j^2}, \quad u_i^2 = \frac{\rho^2 - x_j^2/u_j^2}{\rho^2 - x_j^2} = \frac{x_i^2}{\rho^2 - x_j^2},$$

where  $j \neq i$ . The height difference between  $p_1$  and  $p_2$  is  $\Delta z = \frac{x_i}{u_i} \sqrt{1 - u_i^2}$ .

*Proof* The difference vector is  $p_1 - p_2 = (x_1, x_2, \Delta z)$ . Since this strut has length  $\rho$  and is perpendicular to each  $\gamma_i$ , we get

$$\Delta z^2 + x_1^2 + x_2^2 = \rho^2, \quad \Delta z = \frac{x_i}{u_i} \sqrt{1 - u_i^2}.$$

Simple algebraic manipulations, eliminating  $\Delta z$ , lead to the other equations given.  $\square$

## 8.2 The fully kinked regime

We first consider a clasp constructed of very stiff rope, consisting of circle arcs and line segments (see Figures 5).

**Proposition 8.6** *Let  $C_K$  be the curve in the right half-plane of the  $xz$ -plane consisting of*

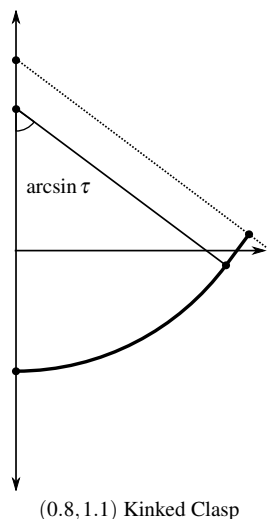
- a circle arc of radius  $\sigma$  of angle  $\arcsin \frac{\tau}{2}$  centered at  $(0, 0, \sigma - 1/2)$ ,
- joined to a line segment in the  $xz$ -plane, where the tip of the second component passes through  $(0, 0, 1/2)$ .

*If  $\sigma \geq 1$ , the corresponding  $2*2$  symmetric curve  $\tilde{C}_K$ , the tip of whose first component lies at the center of the circle arc of the second, is a critical curve for the  $(\tau, \sigma)$ -clasp problem.*

*Proof* We must check that (i)  $\tilde{C}_K$  obeys the endpoint constraints, (ii)  $\tilde{C}_K$  obeys the thickness constraint, and (iii)  $\tilde{C}_K$  is  $\sigma$ -critical. The first is clear from the construction. For the second, we first note that the radius of curvature is always at least  $\sigma$  by construction, so that if the struts have length at least 1, the thickness constraint is satisfied. In fact, by Lemma 8.3 and symmetry, if  $\sigma > 1$  the only strut is the one joining the tip points  $(0, 0, 1/2)$  and  $(0, 0, -1/2)$ . (If  $\sigma = 1$ , there is a family of struts joining each point on each circle arc to the tip of the other component of the clasp.)

To check that our configuration is  $\sigma$ -critical, since the hypotheses are clearly satisfied we may apply the final version of the balance criterion. We let the strut





**Fig. 5** This diagram shows the construction of the fully kinked clasp of Proposition 8.6 with  $(\tau, \sigma) = (0.8, 1.1)$ . The dotted lines are the intersection of two faces of the bounding tetrahedron with the  $xz$ -plane. The entire curved portion of the clasp is a single circle arc of radius  $\sigma$ . We call the  $z$ -intercept of this arc the *tip*. The tip of the other component of the clasp is also marked on the diagram, passing through the  $xz$ -plane at  $(0, 0, 1/2)$ .

measure be an atom of mass  $2\tau$  on the unique strut. The arcs are then balanced against each other by the kink tension function  $\phi$  of Lemma 6.13. On the straight segments,  $T' = 0$  and  $\phi = 0$ , so the balance equation is clearly satisfied. At the endpoints,  $\phi = 0$  and there is no strut force measure, so we require only that the curve be normal to the constraint plane, which is true by construction.  $\square$

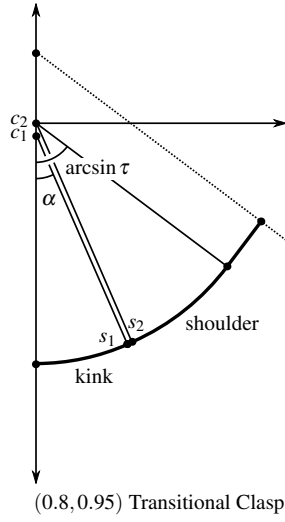
We note that Lemma 6.13 tells us that such a configuration of circle arcs of turning angles  $2\theta_0$  and  $2\theta_1$  and lines is  $\sigma$ -critical as above if and only if  $\sin \theta_0 = \sin \theta_1$ . This means that in addition to the construction above, where  $\theta_0 = \theta_1 \leq \pi/2$ , there are balanced solutions with  $\theta_0 \leq \pi/2 \leq \theta_1$  where a short circle arc balances a longer one, as well as balanced solutions with  $\theta_0 = \theta_1 > \pi/2$ . These are interesting  $\sigma$ -critical curves, but they do not satisfy the boundary conditions of the  $(\tau, \sigma)$ -clasp problems.

### 8.3 The transitional regime

In the transitional regime, the clasp is a circle-line-circle-line curve as in Figure 6.

**Proposition 8.7** Suppose  $\tau \leq 2$ . Let  $C_T$  be the  $C^1$  curve in the right half-plane of the  $xz$ -plane consisting of

- a (kinked) circle arc of angle  $\arcsin \tau/2$  and radius  $\sigma$ ,
- joined by a line segment of length  $\frac{\tau(1-\sigma)}{\sqrt{4-\tau^2}}$  to



**Fig. 6** This diagram shows the construction of the transitional clasp of Proposition 8.7 with  $(\tau, \sigma) = (0.8, 0.95)$ . As before, the dotted lines are the intersection of two faces of the bounding tetrahedron with the  $xz$ -plane. The tip of the second component passes through the origin. With respect to these coordinates, the transitional clasp consists of a lower “kinked” circle arc of radius  $\sigma$  and an upper “shoulder” circle arc of radius 1. The inner arc extends to an angle  $\alpha$  from the  $z$ -axis, while the shoulder extends to angle  $\arcsin \tau$ .

- a circle arc of radius 1 and angle  $\arcsin \tau - \arcsin \tau/2$  (we will refer to this arc as the **shoulder**), with
- a ray attached to the other end of the shoulder.

If

$$1 > \sigma \geq \frac{\sqrt{4 + \tau^2} - 2}{2 - \sqrt{4 - \tau^2}} \quad (4)$$

then this curve exists, and the corresponding  $2 \times 2$  symmetric curve  $\tilde{C}_T$ , the tip of whose second component lies at the center of the shoulder of the first, is a critical curve for the  $(\tau, \sigma)$ -clasp problem.

**Remark 8.8** Since  $\frac{\sqrt{4 + \tau^2} - 2}{2 - \sqrt{4 - \tau^2}} < 1$  for  $\tau \in (0, 1]$ , we see that for each such  $\tau$  the condition (4) is not vacuous.

*Proof* We first show that  $C_T$  exists. Referring to Figure 6, we choose coordinates so that the center of the shoulder arc lies at the origin of the  $xz$ -plane. Then endpoints of the shoulder arc are

$$(\tau, 0, -\sqrt{1 - \tau^2}), \quad s_2 := \left( \frac{\tau}{2}, 0, -\sqrt{1 - \tau^2/4} \right). \quad (5)$$

One endpoint of the segment is  $s_2$ , and the segment has slope

$$m := \frac{\tau}{\sqrt{4 - \tau^2}} \iff \tau = \frac{2m}{\sqrt{1 + m^2}}. \quad (6)$$

Thus the  $x$  and  $z$  coordinates of a point on the segment are related by

$$z = \frac{\tau}{\sqrt{4-\tau^2}} \left( x - \frac{\tau}{2} \right) - \frac{\sqrt{4-\tau^2}}{2}. \quad (7)$$

From the value for the length of the segment given in the Proposition it is easily computed that its other endpoint is

$$s_1 := \left( \frac{\sigma\tau}{2}, 0, \frac{\sigma\tau^2-4}{2\sqrt{4-\tau^2}} \right). \quad (8)$$

This endpoint coincides with one endpoint of the kinked arc of radius  $\sigma$ . Putting  $c_1$  for the center of this arc, the radial vector  $s_1 - c_1$  is parallel to the radial vector  $s_2$  of the shoulder, i.e., makes the angle  $\arcsin \frac{\tau}{2}$  with the vertical. Thus the center of this arc is

$$c_1 := \left( 0, 0, \frac{\sigma\tau^2-4}{2\sqrt{4-\tau^2}} + \sigma \frac{\sqrt{4-\tau^2}}{2} \right) = \left( 0, 0, \frac{2\sigma-2}{\sqrt{4-\tau^2}} \right)$$

and the tip of  $C$  is  $p_0 := (0, 0, z_0)$ , where

$$z_0 := \frac{2\sigma-2}{\sqrt{4-\tau^2}} - \sigma. \quad (9)$$

Next we show that if (4) holds then  $\tilde{C}_T$  has  $\text{Thi}_\sigma \geq 1$ . It is easy to see that its curvature satisfies  $\kappa \leq 1/\sigma$  (since  $\sigma < 1$ ), so we need only show that all the critical pairs have length at least 1. Let us call the two components of the curve  $C$  and  $C^*$ , and put  $p_0^* = (0, 0, 0)$  for the tip point of  $C^*$ .

If  $(p, p^*) \in C \times C^*$  is a critical pair with  $p$  on the kink arc of  $C$ , then by Lemma 8.3 either  $p = p_0$  or else  $p^* = p_0^*$  (or both). In the first case the shoulders of  $C^*$  lie on the boundary of the ball of radius 1 about  $p_0$ , and by elementary geometry the rest of  $C^*$  lies strictly outside it. Therefore any such pair has length at least 1. The same argument with  $C$  and  $C^*$  interchanged yields the same conclusion in the second case.

If  $(p, p^*)$  is a critical pair with  $p$  on the shoulder of  $C$ , then  $p^* = p_0^*$  by Lemma 8.3 again. Hence  $|p - p^*| \geq 1$  by the last paragraph.

By symmetry it remains to consider the case of critical pairs  $(p, p^*)$  where the points lie on the respective straight segments of  $C$  and  $C^*$ . We show that if (4) holds then  $\rho := |p - p^*| \geq 1$ . Put

$$p := (x_1, 0, z_1), \quad q := (0, y_1, z_1^*).$$

By (6), the sine of the angle made by the respective segments with the  $x$ - and  $y$ -axes is  $u := \tau/2$ . Then by Lemma 8.5,

$$x_1^2 = y_1^2 = \frac{\rho^2 u^2}{1+u^2} = \frac{\rho^2 (\frac{\tau}{2})^2}{1+(\frac{\tau}{2})^2} = \frac{\rho^2 \tau^2}{4+\tau^2}. \quad (10)$$

In particular  $p$  and  $p^*$  correspond to one another under the symmetry of the clasp, and the midpoint of the segment  $pp^*$  lies on the horizontal plane equidistant from the

two tips  $p_0, p_0^*$ . Therefore the difference in heights between  $p$  and  $p_0^*$  is equal to the difference in heights between  $p_0$  and  $p^*$ , i.e.,

$$-z_1 = z_1^* - z_0. \quad (11)$$

On the other hand, by Lemma 8.5 the difference in the heights of  $p, p^*$  is

$$\Delta z := z_1^* - z_1 = \frac{x_1}{u} \sqrt{1-u^2} = \frac{x_1}{\tau/2} \sqrt{1-(\tau/2)^2} = \frac{x_1}{\tau} \sqrt{4-\tau^2}. \quad (12)$$

Substituting (9) and solving the system (11), (12) we obtain

$$x_1 = \frac{\tau}{\tau^2+4} \left[ 2 + \sigma \left( 2 - \sqrt{4-\tau^2} \right) \right] \quad (13)$$

and from (10)

$$\rho = \frac{2 + \sigma \left( 2 - \sqrt{4-\tau^2} \right)}{\sqrt{\tau^2+4}}. \quad (14)$$

The thickness condition is violated if and only both  $\rho < 1$  and the points  $p$  lies on the segment of  $C$  (rather than somewhere on the rest of the line it determines). The latter condition is equivalent to the condition that  $x_1$  lie between the  $x$  coordinates of  $s_1$  and  $s_2$ , i.e.,

$$\frac{\tau\sigma}{2} < x_1 < \frac{\tau}{2}$$

in view of (5), (8), or by (13), (10)

$$\frac{\sigma}{2} < \frac{\rho}{\sqrt{\tau^2+4}} < \frac{1}{2}. \quad (15)$$

The second inequality of (15) is a clear consequence of  $\rho < 1$ , which may in turn be expressed as

$$\sigma < \frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}}. \quad (16)$$

Substituting (14), the first inequality of (15) is equivalent to

$$\sigma < \frac{4}{\tau^2 + 2\sqrt{4-\tau^2}}. \quad (17)$$

We claim that the right hand side of (17) dominates that of (16) in the relevant range  $0 \leq \tau \leq 2$ . Putting  $t := \tau^2/4$  this is equivalent to the inequality

$$t + \sqrt{1-t} \leq \frac{1 - \sqrt{1-t}}{\sqrt{1+t}-1} = \frac{(\sqrt{1+t} - \sqrt{1-t}) + (1 - \sqrt{1-t^2})}{t}, \quad 0 \leq t \leq 1. \quad (18)$$

To prove (18), we note

$$\frac{t}{2} \leq 1 - \sqrt{1-t}, \quad 0 \leq t \leq 1, \quad (19)$$

so the left hand side of (18) is dominated by  $1 + \frac{t}{2}$ . On the other hand (19) also yields immediately

$$\frac{t^2}{2} \leq 1 - \sqrt{1 - t^2}, \quad t \leq \sqrt{1 + t} - \sqrt{1 - t}$$

for  $0 \leq t \leq 1$ , so  $1 + \frac{t}{2}$  is dominated by the right hand side of (18) in turn.

Thus (16) is the effective condition. But this is precisely the negation of (4) (assuming we are not in the fully kinked case). So we have now shown that if  $(\tau, \sigma)$  obey our conditions then  $\text{Thi}_\sigma(\tilde{C}_T) \geq 1$ .

Finally we show that the curve is (strongly)  $\sigma$ -critical with the given endpoint constraints by showing it is regularly balanced.

There is a one-parameter family of struts joining each point on the shoulder arcs to the opposite tip. By Lemma 8.2, the strut measure  $ds$  on these struts balances the shoulders. Further, this measure generates a strut force measure of magnitude  $\tau$  at the tip. By Lemma 6.13, this is balanced by a  $\phi$  function on the kink if and only if the angle of the kink is  $\arcsin(\tau/2)$ . But this is true by construction. As before,  $\tilde{C}_T$  is normal to the constraint planes at the endpoints of the arc, so the endpoint conditions of Theorem 5.10 are satisfied as well.

This completes the proof of Proposition 8.7.  $\square$

#### 8.4 The Gehring regime

We have now described the clasp structures in very stiff rope with  $\sigma > \frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}}$ . These are characterized by kinked circle arcs in balance with shoulder arcs. We now jump to the opposite end of the spectrum and describe clasps in very flexible rope with  $\sigma < \sqrt{1-\tau^2}$ . The generic clasp, described in Section 8.5 will combine features from both of these situations.

In [CFK<sup>+</sup>06], we gave a critical clasp structure for the “link-ropelength problem” where the two arcs of the clasp are required to stay 1 unit apart and obey endpoint constraints, but no curvature constraint is enforced. The main result of the present section is the following, which states that the curvature condition is satisfied in the region  $\sigma \leq \sqrt{1-\tau^2}$  of the  $(\tau, \sigma)$ -plane. Except for this condition the statement reproduces Theorem 9.5 of [CFK<sup>+</sup>06], with slight modifications in the notation.

**Theorem 8.9** *Suppose  $\sigma \leq \sqrt{1-\tau^2}$ . Consider the curve  $C_1$  in the  $xz$ -plane given parametrically for  $u \in [-\tau, \tau]$  by*

$$\begin{aligned} x = x_\tau(u) &:= \frac{u\sqrt{1-(\tau-|u|)^2}}{\sqrt{1-u^2(\tau-|u|)^2}}, \\ z = z_\tau(u) &:= \int \frac{\partial z}{\partial x} dx = \int \frac{u}{\sqrt{1-u^2}} \frac{\partial u}{\kappa_\tau(u)}, \end{aligned} \tag{20}$$

where

$$\kappa_\tau(u) := \frac{\sqrt{(1-u^2(\tau-|u|)^2)^3(1-(\tau-|u|)^2)}}{1-(\tau-|u|)^2+(\tau-|u|)|u|(1-u^2)} \tag{21}$$

and the constant of integration for  $z$  is chosen so that

$$z(0) + z(\tau) = -\sqrt{1 - \tau^2}.$$

There is a curve  $C_2$  in the  $yz$ -plane, congruent to  $C_1$  and lying at distance exactly 1 from  $C_1$ , such that  $\tilde{C}_{Ge} := C_1 \cup C_2$  is  $2 * 2$  symmetric, with  $\text{Thi}_\sigma(\tilde{C}_{Ge}) = 1$ , and is critical for the  $(\tau, \sigma)$ -clasp problem.

*Remark 8.10* As described in [CFK<sup>+</sup>06], the parameter  $u$  equals the sine of the angle between the tangent to  $C_1$  and the  $x$ -axis. The function  $\kappa_\tau$  is the curvature. Each point  $(x(u), 0, z(u)) \in C_1$  is connected by two struts of length 1 to symmetrically located points  $(0, \pm x(u^*), -z(u^*)) \in C_2$ , where  $u + u^* = \tau$ . These struts bear a strut measure which balances the curvature measure on each arc of the curve.

Following [CFK<sup>+</sup>06], the parameters  $u, u^*$  as above are said to be **conjugate**. Likewise, a subarc  $A \subset C_1$  corresponding to  $c \leq u \leq d$  is said to be conjugate to the subarcs of  $C_2$  corresponding to  $\tau - d \leq u^* \leq \tau - c$ . In other words the conjugate arcs to  $A$  are precisely the subarcs of  $C_2$  that are joined to  $A$  by struts.

*Proof* The only thing to check is that the curvature function  $\kappa_\tau(u) \leq 1/\sigma$  when  $u \in [0, \tau]$ . To prove it, it will be convenient to define  $\alpha, \beta, \gamma \in [0, \frac{\pi}{2}]$  by

$$\sin \alpha = u, \quad \sin \beta = \tau - \sin \alpha, \quad \sin \gamma = \sin \alpha \sin \beta.$$

Then by (21)

$$\kappa_\tau(u) = \kappa_\tau(\sin \alpha) = \frac{\cos \beta \cos^3 \gamma}{\cos^2 \beta + \sin \gamma \cos^2 \alpha} \leq \frac{\cos^3 \gamma}{\cos \beta} \leq \frac{\cos \gamma}{\cos \beta}. \quad (22)$$

Furthermore

$$\frac{1}{\sigma} \geq \frac{1}{\sqrt{1 - \tau^2}} \geq \frac{1}{\sqrt{1 - \sin^2 \beta}} = \frac{1}{\cos \beta}.$$

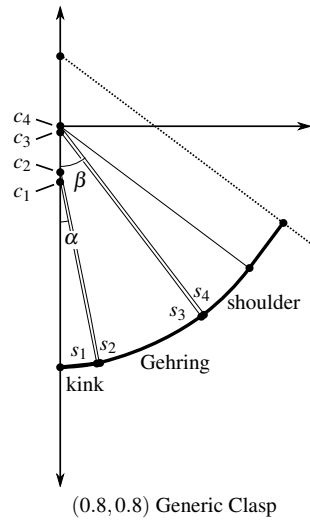
since  $\tau \geq \sin \beta$ . Therefore

$$\frac{1}{\sigma} \geq \frac{1}{\cos \beta} \geq \frac{\cos \gamma}{\cos \beta} \geq \kappa_\tau(u),$$

as desired.  $\square$

## 8.5 The generic regime

We now describe the most complicated clasps. As the stiffness of the curve decreases from the transitional regime, the transitional clasp develops a self-contact in the middle of the straight segment. This contact causes the straight segment to split into two straight segments, with an arc of the Gehring clasp of Theorem 8.9 between them. The kink and shoulder arcs remain, though they become smaller (they will eventually vanish) as the stiffness continues to decrease. These clasps are pictured in Figure 7.



**Fig. 7** This diagram shows the construction of the generic clasp of Proposition 8.11 with  $(\tau, \sigma) = (0.8, 0.8)$ . The dotted lines are the intersection of two faces of the bounding tetrahedron with the  $xz$ -plane. The generic clasp consists of a kinked circle arc of radius  $\sigma$ , a straight segment, an arc of the Gehring clasp, another straight segment, and a “shoulder” circle arc of radius 1. The length of the straight segments is exaggerated on this picture; their true length is close to the width of the lines used to draw the radii.

**Theorem 8.11** Suppose  $\frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}} > \sigma > \sqrt{1-\tau^2}$ .

1. There exists a unique solution  $(\alpha, \beta, \gamma, a, b)$  to the system of equations

$$\sin \alpha + \sin \beta = \tau, \quad (23a)$$

$$\sin \gamma = \sin \alpha \sin \beta, \quad (23b)$$

$$\frac{b}{\sin \beta} = a \sin \alpha + \sigma(1 - \cos \alpha) \quad (23c)$$

$$b \cos \beta = \sin \beta - \frac{\cos \alpha \sin \beta}{\cos \gamma}, \quad (23d)$$

$$a \cos \alpha = \frac{\sin \alpha \cos \beta}{\cos \gamma} - \sigma \sin \alpha, \quad (23e)$$

with  $\alpha, \beta, \gamma \in [0, \pi/2]$ ,  $\sin \alpha \leq \tau/2$ , and  $a, b > 0$ .

2. Given this solution, there is a  $C^1$  curve  $C_\Gamma$  in the right half-plane of the  $xz$ -plane as shown in Figure 7, consisting of the following pieces joined in succession:
- a kinked circle arc of angle  $\alpha$ , meeting the  $z$ -axis orthogonally
  - a straight segment of length  $a$
  - the arc  $\alpha \leq u \leq \beta$  arc of the Gehring clasp of Theorem 8.9
  - a straight segment of length  $b$
  - a “shoulder” circle arc of radius 1 from angle  $\beta$  to angle  $\arcsin \tau$ .

Furthermore, if we denote by  $\tilde{C}_\Gamma$  the corresponding  $2*2$ -symmetric curve, the tip of whose second component lies at the center of the shoulder arc of the first, then the Gehring arcs of the two components of  $\tilde{C}_\Gamma$  are conjugate.

3.  $\text{Thi}_\sigma(\tilde{C}_\Gamma) = 1$ .

4.  $\tilde{C}_\Gamma$  is critical for the  $(\tau, \sigma)$ -clasp problem.

*Proof* (1): Let us change our point of view by taking  $\tau$  as given, and viewing (23) as a 1-parameter family of systems in the unknowns  $\sigma, \beta, \gamma, a, b$  as the parameter  $\alpha$  varies from 0 to  $\arcsin \frac{\tau}{2}$ . It is clear that (23a), (23b), (23d) determine  $\beta, \gamma, b$  uniquely, with  $b > 0$  since

$$\cos \gamma = \sqrt{1 - \sin^2 \gamma} = \sqrt{1 - \sin^2 \alpha \sin^2 \beta} > \sqrt{1 - \sin^2 \alpha} = \cos \alpha. \quad (24)$$

Solving (23c), (23e) for  $a, \sigma$  and substituting the value for  $b$  arising from (23d), we obtain

$$\sigma = \frac{\sin^2 \alpha \cos^2 \beta + \cos^2 \alpha - \cos \alpha \cos \gamma}{(1 - \cos \alpha) \cos \beta \cos \gamma} = \frac{\cos \gamma - \cos \alpha}{(1 - \cos \alpha) \cos \beta} = \frac{(1 + \cos \alpha) \cos \beta}{\cos \gamma + \cos \alpha} \quad (25)$$

and

$$a = \tan \alpha \cos \beta \left( \frac{1}{\cos \gamma} - \frac{1 + \cos \alpha}{\cos \gamma + \cos \alpha} \right) = \tan \alpha \cos \beta \frac{\cos \alpha (1 - \cos \gamma)}{\cos \gamma (\cos \gamma + \cos \alpha)} > 0. \quad (26)$$

Thus we may show that (23) is uniquely solvable in the original sense, with  $\sigma$  given and  $\alpha$  unknown, by establishing that (25) expresses  $\sigma$  as a continuous strictly increasing function of  $\alpha$ , with  $\sigma(\arcsin(\tau/2)) = \frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}}$  and  $\sigma(0) = \sqrt{1-\tau^2}$ . The latter relations may be verified directly, and continuity of  $\sigma$  is trivial. To prove that  $\sigma$  is strictly increasing, since  $\sin \alpha$  and  $\sin \gamma = \sin \alpha(\tau - \sin \alpha)$  are both increasing in the range  $0 \leq \sin \alpha \leq \frac{\tau}{2}$ , it is clear that both  $\cos \alpha$  and  $\cos \gamma$  are decreasing functions of  $\alpha$ . Thus it remains only to show that the numerator  $(1 + \cos \alpha) \cos \beta$  of (25) is increasing as a function of  $u := \sin \alpha \in [0, \tau/2]$ . Since

$$\frac{d}{du} \cos \alpha = -\tan \alpha, \quad \frac{d}{du} \sin \beta = -1, \quad \frac{d}{du} \cos \beta = \tan \beta,$$

we compute

$$\frac{d}{du} (1 + \cos \alpha) \cos \beta = -\tan \alpha \cos \beta + (1 + \cos \alpha) \tan \beta > \tan \beta - \tan \alpha.$$

But  $\sin \alpha + \sin \beta = \tau$  and  $\sin \alpha < \tau/2$ , so

$$\sin \beta > \sin \alpha \implies \beta > \alpha \implies \tan \beta > \tan \alpha.$$

(2) Letting  $x(u) = x_\tau(u)$  denote the parametrization of the Gehring arc given in (20), the  $x$ -coordinates of the two endpoints of this arc are

$$x(\sin \alpha) = \frac{\sin \alpha \cos \beta}{\cos \gamma}, \quad x(\sin \beta) = \frac{\cos \alpha \sin \beta}{\cos \gamma}$$



by (23a) and (20). On the other hand the  $x$ -coordinates of the inner endpoints of the kink and the shoulder arcs are given by  $\sigma \sin \alpha, \sin \beta$  respectively. Since by part (1)

$$\begin{aligned} a \cos \alpha &= x(\sin \alpha) - \sigma \sin \alpha = \frac{\sin \alpha \cos \beta}{\cos \gamma} - \sigma \sin \alpha > 0, \\ b \cos \beta &= \sin \beta - x(\sin \beta) = \sin \beta - \frac{\cos \alpha \sin \beta}{\cos \gamma} > 0, \end{aligned}$$

we may interpolate straight segments of lengths  $a, b$  between the kink and the Gehring arc, and between the Gehring arc and the shoulder, respectively, to obtain a  $C^1$  curve  $C_\Gamma$  as described.

Next we show that the Gehring arcs of the two components of  $\tilde{C}_\Gamma$  are conjugate to each other provided the components are situated with the tip of one at the center of the shoulder of the other. Referring to Figure 7, this is to say that the point  $c_3$  is the projection to the  $xz$ -plane of the point  $s_2^*$  of the other component that corresponds to  $s_2$ . If the center of the shoulder arc (which is the tip of the other component) is the origin then the  $z$ -coordinate of  $c_3$  is clearly  $b/\sin \beta$ . On the other hand, since the two components are congruent the  $z$ -coordinate of  $s_2^*$  equals the difference in the  $z$ -coordinates of  $s_2$  and the tip of  $C_\Gamma$ . Equating these two,

$$\frac{b}{\sin \beta} = a \sin \alpha + \sigma(1 - \cos \alpha)$$

which is (23c).

(3): We show first that the curvature of  $C_\Gamma$  is no more than  $1/\sigma$ . The kink, shoulder, and straight segments clearly obey this bound, so we need only check the Gehring clasp arc. We parametrize this arc by  $u \in [\sin \alpha, \sin \beta]$  as in Theorem 8.9. Viewing  $\sigma = \sigma(\alpha)$  as in (25) above, we must check that

$$\kappa_\tau(u) \leq 1/\sigma(\alpha) \quad (27)$$

on this interval. We carry this out for the two subintervals  $[\sin \alpha, \tau/2], [\tau/2, \sin \beta]$  separately.

Since  $\sigma(\alpha)$  is strictly increasing in  $\alpha$  for  $\sin \alpha \in [0, \tau/2]$ , for  $u$  in this range we have  $1/\sigma(u) \leq 1/\sigma(\alpha)$  and it suffices to show  $\kappa_\tau(u) \leq 1/\sigma(u)$ . Define  $\alpha'$  by  $\sin \alpha' = u$ , and  $\beta', \gamma'$  analogously to (23a) and (23b). Then

$$\kappa_\tau(u) = \kappa_\tau(\sin \alpha') = \frac{\cos \beta' \cos^3 \gamma'}{\cos^2 \beta' + \sin \gamma' \cos^2 \alpha'} \leq \frac{\cos \beta' \cos^3 \gamma'}{\cos^2 \beta'} \leq \frac{\cos \gamma'}{\cos \beta'}.$$

On the other hand, by (25)

$$\frac{1}{\sigma(u)} = \frac{\cos \gamma' + \cos \alpha'}{(1 + \cos \alpha') \cos \beta'}$$

and (27) follows easily.

To cover the range  $u \in [\tau/2, \sin \beta]$  it suffices to prove that  $\kappa_\tau(u^*) \leq 1/\sigma(u)$  for  $u \in [\sin \alpha, \tau/2]$ , where  $u + u^* = \tau$  (i.e.,  $u, u^*$  are conjugate). Since replacing  $u$  by  $u^*$  exchanges the variables  $\alpha'$  and  $\beta'$  and leaves  $\gamma'$  unchanged,

$$\kappa_\tau(u^*) = \frac{\cos \alpha' \cos^3 \gamma'}{\cos^2 \alpha' + \sin \gamma' \cos^2 \beta'} \leq \frac{\cos^3 \gamma'}{\cos \alpha'} \leq \frac{\cos \gamma'}{\cos \alpha'}.$$

On the other hand,

$$\frac{1}{\sigma(u)} = \frac{\cos \gamma' + \cos \alpha'}{(1 + \cos \alpha') \cos \beta'} \geq \frac{\cos \gamma' + \cos \gamma' \cos \alpha'}{(1 + \cos \alpha') \cos \beta'} = \frac{\cos \gamma'}{\cos \beta'}.$$

Now (27) follows from the fact that  $\sin \alpha' \leq \tau/2 \leq \sin \beta'$ .

Next we claim that all critical pairs  $(p, p^*)$  of the distance between the components of  $\tilde{C}_\Gamma$  satisfy  $|p - p^*| \geq 1$ . To simplify the discussion we will put  $C_\Gamma^*$  for the part of the second component lying in the  $y \geq 0$  part of the  $yz$ -plane, and consider only those pairs with  $p \in C_\Gamma, p^* \in C_\Gamma^*$ .

The claim is clearly true if  $p$  lies on the Gehring arc, since in this case  $p^*$  is the conjugate point of the Gehring arc of  $C_\Gamma^*$ .

Note that if  $(p, p^*)$  is a critical pair then the projection of the segment  $pp^*$  to the  $xz$ -plane is a line segment perpendicular to  $C_\Gamma$  at  $p$  and with the other endpoint on the  $z$ -axis. Now if we denote by  $z^*(p)$  the  $z$ -intercept of the normal line through  $C_\Gamma$  at  $p$ , then  $z^*$  is an increasing function of the  $x$ -coordinate of  $p$ . (This is obvious for the circle arcs and line segments, and true for the Gehring arc by construction.)

By Lemma 8.3, if  $p$  lies on the shoulder arc or the kink then  $p^*$  is the tip of  $C_\Gamma^*$ . In the shoulder case  $|p - p^*| = 1$  by construction. To handle the kink case we note that every point of  $C_\Gamma$  lies at distance  $\geq 1$  from the tip of  $C_\Gamma^*$ : otherwise  $C_\Gamma$  crosses the circle of radius 1 about the origin in the  $xz$ -plane at some point  $p$ . Since the slope of  $C_\Gamma$  must be less than the slope of the circle at this point, it follows that  $z^*(p) > z^*(s_4) = 0$ . But  $z^*(p) \leq 0$  by monotonicity.

By monotonicity of  $z^*$  again, and symmetry, it remains only to consider the case where  $p \in s_1 s_2$  and  $p^* \in s_3^* s_4^*$ . However, since the lines generated by these segments are skew, there is at most one such critical pair. This pair is  $p = s_2, p^* = s_3^*$ , i.e., the common endpoints of the segments and the Gehring arcs.

(4): We will show  $\tilde{C}_\Gamma$  is regularly balanced.

There is a one-parameter family of struts joining each point on the shoulder arcs to the opposite tip. By Lemma 8.2, the strut measure  $ds$  on these struts balances the shoulders. Further, this measure generates a strut force measure of magnitude  $\tau$  at the tip. By Lemma 6.13, this is balanced by a  $\phi$  function on the kink if and only if the angle of the kink is  $\arcsin(\tau/2)$ . But this is true by (23a). The straight segments bear no strut force and have  $T' = 0$ , so they obey the balance equation as well. Further, the Gehring arcs obey the balance equation by construction.

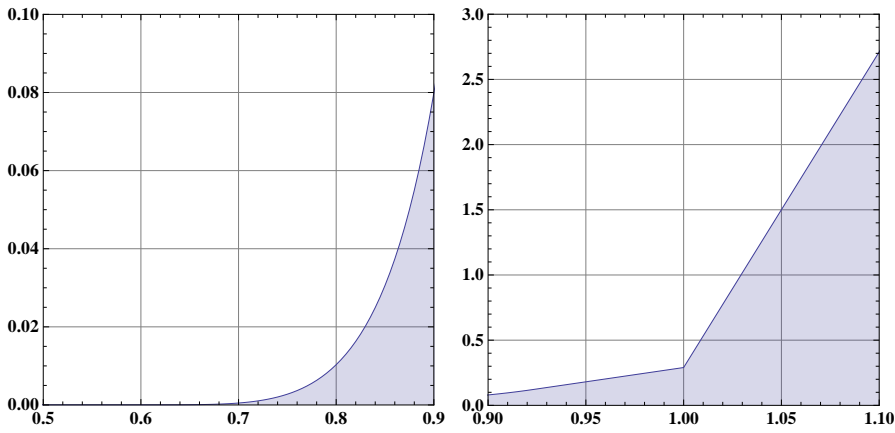
As before,  $\tilde{C}_\Gamma$  is normal to the constraint planes at the endpoints of the arc, so the endpoint conditions of Theorem 5.10 are satisfied as well.

This completes the proof of Theorem 8.11.  $\square$

### 8.6 Geometry of the tight clasps

To compare the length of various clasps with the same  $\tau$ , but different  $\sigma$  without fixing a particular bounding tetrahedron, we define the **excess length** of a  $(\tau, \sigma)$  clasp to be the difference between the length of the clasp and 4 times the inradius of the bounding tetrahedron. As  $\sigma$  increases, we are strengthening the curvature constraint, and we expect the excess length to increase.

While the excess length of the kinked and transitional clasps can be computed exactly, the length of the Gehring clasp (and the generic clasp, which includes a Gehring arc) is only known as the solution of a certain hyperelliptic integral [CFK<sup>+</sup>06]. We constructed all of our clasps numerically, checking the thickness and curvature of each with *octrope* [AC05], and computing the excess length by numerical integration. The results are shown in Figure 8 shows the relationship between excess length and  $\sigma$  for the clasps with  $\tau = 0.8$ .

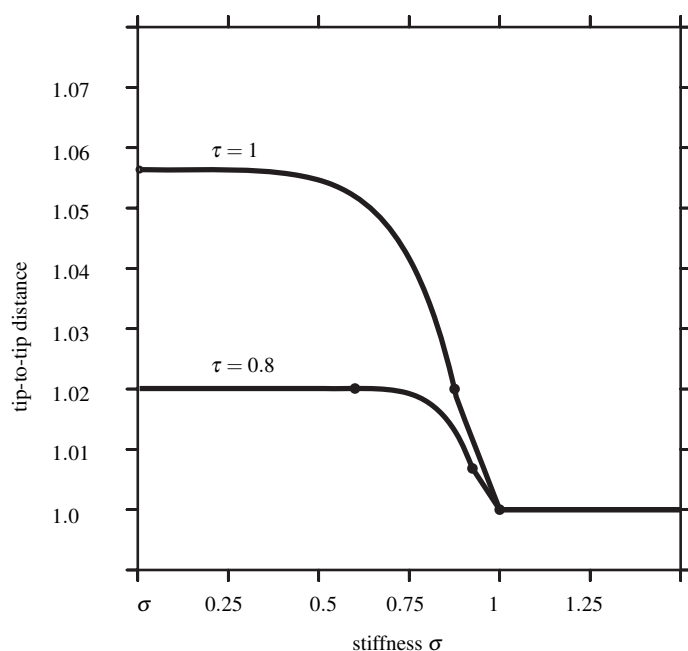


**Fig. 8** This set of graphs shows the relationship between excess length and  $\sigma$  for our solutions to the  $(\tau, \sigma)$  clasp problem with  $\tau = 0.8$ . The excess length of the Gehring clasp for this value of  $\tau$  is 2.10308. The graph shows the increase in excess length as a function of  $\sigma$ , expressed as a percentage of the excess length of the Gehring clasp. For example, at  $\sigma = 1.05$ , our (fully kinked) solution is a clasp 1.5% longer than the Gehring clasp. The boundary between the Gehring regime and the generic regime occurs at  $\sigma = 0.6$ , the boundary between the generic and transitional regimes at  $\sigma = 0.927$  and the boundary between the transitional and kinked regimes at  $\sigma = 1$ . The graphs show that as  $\sigma$  decreases, relaxing the curvature constraint, the clasp is able to become shorter. Note that decreasing  $\sigma$  below the Gehring/generic boundary has no further effect, as the Gehring clasps for  $\sigma < 0.6$  are all the same curve (the curvature constraint is not active). Note also that the excess length function appears to be  $C^1$  across the Gehring/generic and generic/transitional boundaries, but clearly has a corner at the transitional/kinked boundary.

How much length can be saved by relaxing the curvature constraint? The excess length of the kinked  $\sigma = 1$  clasp with  $\tau = 0.8$  is 2.109180872 and that of the Gehring clasp with  $\sigma = 1/2$  and  $\tau = 0.8$  is 2.103080861; these differ by about 0.3%. For  $\tau = 1 - 10^{-9}$ , the excess length of the kinked  $\sigma = 1$  clasp is 4.28318530 and that of the generic  $\sigma = 1/2$  clasp is 4.26309458; these differ by about 0.46%. We can compare this to the Gehring ( $\sigma = 0$ ) clasp excess length of 4.262897, which is about

0.5% less than that of the  $\sigma = 1$  clasp. We can see from this example, and from the graphs in Figure 8 that very little length is saved over the generic regime.

One of the most striking features of the Gehring clasp was a small gap between the two tubes. This gap formed a small chamber between the two tubes as they were pulled together. We have already seen that the same gap exists in the generic solutions, as we showed above that the tip-to-tip distance was greater than 1. In fact, the tip-to-tip distance is monotonic in  $\sigma$  for each value of  $\tau$ , as we see in Figure 9. For smaller values of  $\tau$ , the maximum tip-to-tip distance decreases as well, reaching 1 only for the trivial  $\tau = 0$  clasp. The maximum tip-to-tip distance, about 1.05653 times the tube diameter, occurs at the Gehring  $(1, 0)$ -clasp. The generic  $(1, 1)$  clasp still has a tip-to-tip distance about 1.05482 times the tube diameter.



**Fig. 9** This graph shows the tip-to-tip distance for the  $\tau = 1$  (upper curve) and  $\tau = 0.8$  (lower curve). We can see that in all the kinked clasps ( $\sigma \geq 1$ ) the tips are in contact, as the tip-to-tip distance is 1. As the stiffness decreases, the force exerted by the shoulder arcs pushes the tips apart, creating a gap between the tubes. We mark the transition between the kinked, transitional, generic, and Gehring regimes with small dots. We can see that the gap size increases monotonically as  $\sigma$  decreases until the transition to the Gehring clasp regime. At that point, the curvature constraint is no longer active and further decreases of  $\sigma$  do not change the curve or the tip-to-tip distance.

**Acknowledgements** We gratefully acknowledge helpful conversations with many colleagues, including Elizabeth Denne, Oscar Gonzalez, and Heiko von der Mosel. Special thanks go to Nancy Wrinkle for various contributions to this project. Some of the figures were prepared with POV-ray, Inkscape and Mathematica. This work was partially supported by the NSF through grants DMS-02-04826 (to Cantarella and Fu) and DMS-10-07580 (to Fu).

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